1a $[\frac{x^2}{2!}]' = [\frac{x^2}{2\cdot 1}]' = [\frac{1}{2}x^2]' = \frac{1}{2}\cdot 2x = x$ en $[\frac{x^3}{3!}]' = [\frac{x^3}{3\cdot 2\cdot 1}]' = [\frac{1}{6}x^3]' = \frac{1}{6}\cdot 3x^2 = \frac{1}{2}x^2 = \frac{x^2}{2!}$.

1b $[\frac{x^n}{n!}]' = [\frac{x^n}{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 1}]' = \frac{n\cdot x^{n-1}}{n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 1} = \frac{x^{n-1}}{(n-1)\cdot(n-2)\cdot\ldots\cdot 1} = \frac{x^{n-1}}{(n-1)!}$.

1c $[e^x]' = [1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots]' = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x$. Dus $f(x) = e^x$ geeft $f'(x) = e^x$.

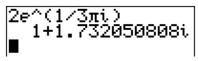
2a $f(x) = \sin(x) \Rightarrow f(0) = \sin(0) = 0.$
$f(x) = \sin(x) \Rightarrow f'(x) = \cos(x) \Rightarrow f'(0) = \cos(0) = 1.$
$f'(x) = \cos(x) \Rightarrow f''(x) = -\sin(x) \Rightarrow f''(0) = -\sin(0) = 0.$
$f''(x) = -\sin(x) \Rightarrow f'''(x) = -\cos(x) \Rightarrow f'''(0) = -\cos(0) = -1.$
$f'''(x) = -\cos(x) \Rightarrow f''''(x) = \sin(x) \Rightarrow f''''(0) = \sin(0) = 0$ enzovoort.
Dus $\sin(x) = 0 + 1\cdot x + 0\cdot\frac{x^2}{2!} - 1\cdot\frac{x^3}{3!} + 0\cdot\frac{x^4}{4!} + 1\cdot\frac{x^5}{5!} + 0\cdot\frac{x^6}{6!} - 1\cdot\frac{x^7}{7!} + \ldots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$

2b $g(x) = \cos(x) \Rightarrow g(0) = \cos(0) = 1.$
$g(x) = \cos(x) \Rightarrow g'(x) = -\sin(x) \Rightarrow g'(0) = -\sin(0) = 0.$
$g'(x) = -\sin(x) \Rightarrow g''(x) = -\cos(x) \Rightarrow g''(0) = -\cos(0) = -1.$
$g''(x) = -\cos(x) \Rightarrow g'''(x) = \sin(x) \Rightarrow g'''(0) = \sin(0) = 0.$
$g'''(x) = \sin(x) \Rightarrow g''''(x) = \cos(x) \Rightarrow g''''(0) = \cos(0) = 1$ enzovoort.
Dus $\cos(x) = 1 + 0\cdot x - 1\cdot\frac{x^2}{2!} + 0\cdot\frac{x^3}{3!} + 1\cdot\frac{x^4}{4!} + 0\cdot\frac{x^5}{5!} - 1\cdot\frac{x^6}{6!} + 0\cdot\frac{x^7}{7!} + \ldots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$

2c $f'(x) = [\sin(x)]' = [x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots]' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots = \cos(x) = g(x).$
$g'(x) = [\cos(x)]' = [1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots]' = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \ldots$
$= -(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots) = -\sin(x) = -f(x).$

3a Voor een complex getal kennen we al de notatie $z = a + bi$ waarbij $a = \text{Re}(z)$ en $b = \text{Im}(z)$ en ook de notatie $z = r(\cos(\varphi) + i\sin(\varphi))$ waarbij $r = |z|$ en $\varphi = \arg(z)$.
- $a = \text{Re}(z) = 1$
- $b = \text{Im}(z) \approx 1{,}732$ (je herkent $\sqrt{3} \approx 1{,}732$)

$\Rightarrow 2e^{\frac{1}{3}\pi i} = 1 + i\sqrt{3}.$



- $r = |z| = 2$
- $\varphi = \arg(z) = 60° = \frac{1}{3}\pi$ rad.

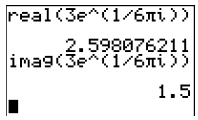
$\Rightarrow 2e^{\frac{1}{3}\pi i} = 2(\cos(60°) + i\sin(60°)).$

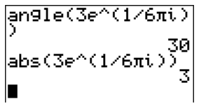
3b De GR geeft:
- $a = \text{Re}(z) \approx 2{,}598$ (herken je $1\frac{1}{2}\sqrt{3}$?)
- $b = \text{Im}(z) = 1{,}5$

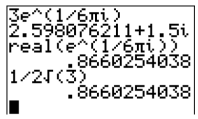
$\Rightarrow 2e^{\frac{1}{6}\pi i} = 1\frac{1}{2}\sqrt{3} + 1{,}5i.$

- $r = |z| = 3$
- $\varphi = \arg(z) = 30° = \frac{1}{6}\pi$ rad.

$\Rightarrow 3e^{\frac{1}{6}\pi i} = 3(\cos(30°) + i\sin(30°)).$







4a $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots$ geeft (vervang x door ix)
$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \ldots$
$= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \frac{i^6x^6}{6!} + \frac{i^7x^7}{7!} + \ldots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \ldots$

4b $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ en $i\sin(x) = i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) = ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \ldots$
Dit geeft $\cos(x) + i\sin(x) = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \ldots = e^{ix}$ (zie 4a).

5a $4e^{\frac{1}{4}\pi i} = 4\left(\cos(\frac{1}{4}\pi) + i\sin(\frac{1}{4}\pi)\right) = 4\left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) = 2\sqrt{2} + 2i\sqrt{2}.$

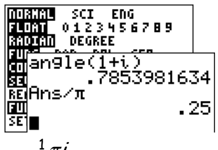
5b $\sqrt{3}\cdot e^{\frac{1}{6}\pi i} = \sqrt{3}\left(\cos(\frac{1}{6}\pi) + i\sin(\frac{1}{6}\pi)\right) = \sqrt{3}\left(\frac{1}{2}\sqrt{3} + \frac{1}{2}i\right) = \frac{3}{2} + \frac{1}{2}i\sqrt{3}.$

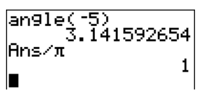
5c $e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1 + 0\cdot i = -1.$

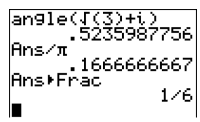
6a $|1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$ en $\text{Arg}(1+i) = \frac{1}{4}\pi \Rightarrow 1 + i = \sqrt{2}\cdot e^{\frac{1}{4}\pi i}$.

6b $|-5| = 5$ en $\text{Arg}(-5) = \pi \Rightarrow -5 = 5e^{\pi i}$.

6c $|\sqrt{3} + i| = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2$ en $\text{Arg}(\sqrt{3} + i) = \frac{1}{6}\pi \Rightarrow \sqrt{3} + i = 2e^{\frac{1}{6}\pi i}$.



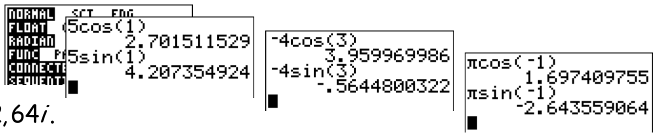




7a $5e^{i} = 5e^{1i} = 5(\cos(1) + i\sin(1)) \approx 2{,}70 + 4{,}21i.$

7b $-4e^{3i} = -4(\cos(3) + i\sin(3)) \approx 3{,}96 - 0{,}56i.$

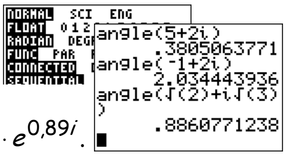
7c $\pi e^{-i} = \pi e^{-1i} = \pi(\cos(-1) + i\sin(-1)) \approx 1{,}70 - 2{,}64i.$



8a $|5+2i| = \sqrt{5^2 + 2^2} = \sqrt{29}$ en $\text{Arg}(5+2i) \approx 0{,}38 \Rightarrow 5 + 2i \approx \sqrt{29} \cdot e^{0{,}38i}.$

8b $|-1+2i| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$ en $\text{Arg}(-1+2i) \approx 2{,}03 \Rightarrow -1 + 2i \approx \sqrt{5} \cdot e^{2{,}03i}.$

8c $\left|\sqrt{2} + i\sqrt{3}\right| = \sqrt{\sqrt{2}^2 + \sqrt{3}^2} = \sqrt{5}$ en $\text{Arg}(\sqrt{2} + i\sqrt{3}) \approx 0{,}89 \Rightarrow \sqrt{2} + i\sqrt{3} \approx \sqrt{5} \cdot e^{0{,}89i}.$

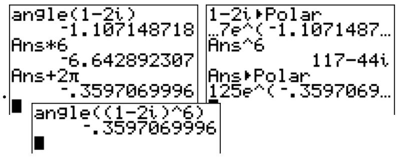


9a $\frac{e^{ix} + e^{-ix}}{2} = \frac{\cos(x) + i\sin(x) + \cos(-x) + i\sin(-x)}{2} = \frac{\cos(x) + i\sin(x) + \cos(x) - i\sin(x)}{2} = \frac{2\cos(x)}{2} = \cos(x).$

9b $\frac{e^{ix} - e^{-ix}}{2i} = \frac{\cos(x) + i\sin(x) - \cos(-x) - i\sin(-x)}{2i} = \frac{\cos(x) + i\sin(x) - \cos(x) + i\sin(x)}{2i} = \frac{2i\sin(x)}{2i} = \sin(x).$

10a $|1-2i| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$ en $\text{Arg}(1-2i) \approx -1{,}107 \Rightarrow 1 - 2i \approx \sqrt{5} \cdot e^{-1{,}107i}.$

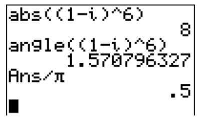
10b $|1-2i|^6 = \left(\sqrt{5} \cdot e^{-1{,}107\ldots i}\right)^6 = \left(\sqrt{5}\right)^6 \cdot \left(e^{-1{,}107\ldots i}\right)^6 \approx (5)^3 \cdot \left(e^{-6{,}643i}\right) = 125 \cdot e^{-6{,}643i}.$



10c De formule van de Moivre is: $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx).$

Dus volgens deze formule is: $z^6 \approx \left(\sqrt{5} \cdot e^{-1{,}107\ldots i}\right)^6 = \left(\sqrt{5}\right)^6 \cdot \left(e^{-1{,}107\ldots i}\right)^6 = 125 \cdot (\cos(-1{,}107\ldots) + i\sin(-1{,}107\ldots))^6$

$= 125 \cdot (\cos(6 \cdot -1{,}107\ldots) + i\sin(6 \cdot -1{,}107\ldots)) \approx 125 \cdot (\cos(6{,}643) + i\sin(6{,}643)).$



11a $1 - i = \sqrt{2} \cdot e^{-\frac{1}{4}\pi i} \Rightarrow (1-i)^6 = \left(\sqrt{2} \cdot e^{-\frac{1}{4}\pi i}\right)^6 = 2^3 \cdot e^{6 \cdot -\frac{1}{4}\pi i} = 8e^{-\frac{3}{2}\pi i}$ of $8e^{\frac{1}{2}\pi i}.$

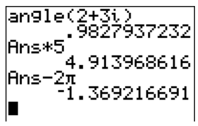
11b $1 + i = \sqrt{2} \cdot e^{\frac{1}{4}\pi i}$ en $3 + 3i = 3\sqrt{2} \cdot e^{-\frac{1}{4}\pi i} \Rightarrow (1+i) \cdot (3-3i) = \sqrt{2} \cdot e^{\frac{1}{4}\pi i} \cdot 3\sqrt{2} \cdot e^{-\frac{1}{4}\pi i} = 6 \cdot e^{\frac{1}{4}\pi i + -\frac{1}{4}\pi i} = 6 \cdot e^0 = 6 \cdot 1 = 6.$

11c $4 + 4i\sqrt{3} = 8e^{\frac{1}{3}\pi i}$ en $\sqrt{3} - i = 2e^{-\frac{1}{6}\pi i} \Rightarrow \frac{4 + 4i\sqrt{3}}{\sqrt{3} - i} = \frac{8e^{\frac{1}{3}\pi i}}{2e^{-\frac{1}{6}\pi i}} = 4e^{\frac{1}{3}\pi i - -\frac{1}{6}\pi i} = 4e^{\frac{1}{2}\pi i}.$

11d $5 + 5i = \sqrt{50} \cdot e^{\frac{1}{4}\pi i} \Rightarrow (5+5i)^4 = \left(\sqrt{50} \cdot e^{\frac{1}{4}\pi i}\right)^4 = 50^2 \cdot e^{\pi i} = 2\,500 \cdot -1 = -2\,500.$

11e $2\sqrt{3} + 2i = 4e^{\frac{1}{6}\pi i}$ en $2 - 2i\sqrt{3} = 4e^{-\frac{1}{3}\pi i} \Rightarrow \left(2\sqrt{3} + 2i\right) \cdot \left(2 - 2i\sqrt{3}\right) = 4e^{\frac{1}{6}\pi i} \cdot 4e^{-\frac{1}{3}\pi i} = 16e^{-\frac{1}{6}\pi i}.$

11f $3\sqrt{3} - 3i = 6e^{-\frac{1}{6}\pi i}$ en $2\sqrt{3} + 2i = 4e^{\frac{1}{6}\pi i} \Rightarrow \frac{(3\sqrt{3} - 3i)^2}{2\sqrt{3} + 2i} = \frac{(6e^{-\frac{1}{6}\pi i})^2}{4e^{\frac{1}{6}\pi i}} = \frac{36e^{-\frac{1}{3}\pi i}}{4e^{\frac{1}{6}\pi i}} = 9e^{-\frac{1}{2}\pi i}.$

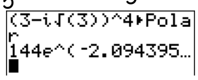


12a $2 + 3i = \sqrt{13} \cdot e^{0{,}983\ldots i} \Rightarrow (2+3i)^5 = \left(\sqrt{13} \cdot e^{0{,}983\ldots i}\right)^5 \approx 13^2 \cdot \sqrt{13} \cdot e^{4{,}91i} \approx 169 \cdot \sqrt{13} \cdot e^{-1{,}37i}.$

12b $(1+4i) \cdot (4-i) = 4 - i + 16i - 4i^2 = 8 + 15i \approx 17 \cdot e^{1{,}08i}.$

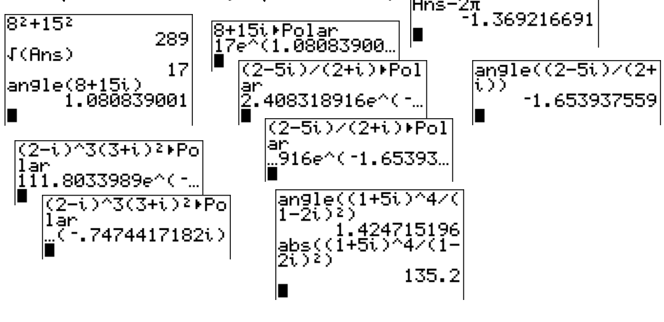
12c $\frac{2-5i}{2+i} \approx \frac{\sqrt{29}}{\sqrt{5}} e^{-1{,}65i} = \frac{\sqrt{29}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} e^{-1{,}65i} = \frac{1}{5}\sqrt{145} e^{-1{,}65i}.$

12d $\left(3 - i\sqrt{3}\right)^4 \approx 144 \cdot e^{-2{,}09i}.$



12e $(2-i)^3 \cdot (3+i)^2 \approx \sqrt{5}^3 \cdot \sqrt{10}^2 \cdot e^{-0{,}75i} = 50\sqrt{5} \cdot e^{-0{,}75i}.$

12f $\frac{(1+5i)^4}{(1-2i)^2} \approx \frac{\sqrt{26}^4}{\sqrt{5}^2} e^{1{,}42i} = \frac{26 \cdot 26}{5} e^{1{,}42i} = 135\frac{1}{5} e^{1{,}42i}.$



13a $e^{\frac{1}{4}\pi i} + \sqrt{2} + i\sqrt{2} = e^{\frac{1}{4}\pi i} + 2e^{\frac{1}{4}\pi i} = 1e^{\frac{1}{4}\pi i} + 2e^{\frac{1}{4}\pi i} = 3e^{\frac{1}{4}\pi i}.$

13b $1 + i\sqrt{3} - e^{\frac{1}{3}\pi i} = 2e^{\frac{1}{3}\pi i} - e^{\frac{1}{3}\pi i} = 2e^{\frac{1}{3}\pi i} - 1e^{\frac{1}{3}\pi i} = e^{\frac{1}{3}\pi i}.$

13c $(2\sqrt{3} - 2i)^2 - 10e^{\frac{5}{3}\pi i} = (4e^{-\frac{1}{6}\pi i})^2 - 10e^{-\frac{1}{3}\pi i} = 16e^{-\frac{1}{3}\pi i} - 10e^{-\frac{1}{3}\pi i} = 6e^{-\frac{1}{3}\pi i}.$

13d $\frac{(1+i)^6}{4e^{\pi i}} = \frac{(\sqrt{2}e^{\frac{1}{4}\pi i})^6}{4e^{\pi i}} = \frac{8e^{1\frac{1}{2}\pi i}}{4e^{\pi i}} = 2e^{\frac{1}{2}\pi i}.$

14a $z_1 = r_1 \cdot e^{i\varphi_1} = r_1\left(\cos(\varphi_1) + i\sin(\varphi_1)\right) \Rightarrow |z_1| = r_1$

$z_2 = r_2 \cdot e^{i\varphi_2} = r_2\left(\cos(\varphi_2) + i\sin(\varphi_2)\right) \Rightarrow |z_2| = r_2$. Dus $|z_1| \cdot |z_2| = r_1 r_2$ **(1).**

$z_1 \cdot z_2 = r_1\left(\cos(\varphi_1) + i\sin(\varphi_1)\right) \cdot r_2\left(\cos(\varphi_2) + i\sin(\varphi_2)\right)$

$= r_1 r_2\left(\cos(\varphi_1) + i\sin(\varphi_1)\right) \cdot \left(\cos(\varphi_2) + i\sin(\varphi_2)\right)$

$= r_1 r_2\left(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right)$ Dus $|z_1 \cdot z_2| = r_1 r_2$ **(2).** Uit **(1)** en **(2)** volgt $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

14b $\dfrac{z_1}{z_2} = \dfrac{r_1 \cdot e^{i\varphi_1}}{r_2 \cdot e^{i\varphi_2}} = \dfrac{r_1}{r_2} \cdot \dfrac{e^{i\varphi_1}}{e^{i\varphi_2}} = \dfrac{r_1}{r_2} \cdot e^{i(\varphi_1 - \varphi_2)} = \dfrac{r_1}{r_2} \cdot \left(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)\right)$. Dus $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2}$ **(1).**

Ook geldt: $|z_1| = r_1$ en $|z_2| = r_2$. Dus $\dfrac{|z_1|}{|z_2|} = \dfrac{r_1}{r_2}$ **(2).** Uit **(1)** en **(2)** volgt $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$.

15a $\left(e^{i\varphi}\right)^n = e^{i\varphi n} = e^{in\varphi} = \cos(n\varphi) + i\sin(n\varphi)$.

$\left(e^{i\varphi}\right)^n = \left(\cos(\varphi) + i\sin(\varphi)\right)^n = \cos(n\varphi) + i\sin(n\varphi)$ en dat is de stelling van De Moivre.

15b $\overline{e^{i\varphi}} = \overline{\cos(\varphi) + i\sin(\varphi)} = \cos(\varphi) - i\sin(\varphi) = \cos(-\varphi) + i\sin(-\varphi) = e^{-i\varphi}$.

16a $z^3 = -8$ heeft drie oplossingen, dus je kunt -8 noteren met drie argumenten die telkens 2π verschillen.

$\left|z^3\right| = |-8| = 8$ en $\text{Arg}(z^3) = \text{Arg}(-8) = \pi$.

Dus $z^3 = 8\left(\cos(\pi) + i\sin(\pi)\right) \;\vee\; z^3 = 8\left(\cos(3\pi) + i\sin(3\pi)\right) \;\vee\; z^3 = 8\left(\cos(5\pi) + i\sin(5\pi)\right)$.

16b $z = \sqrt[3]{8}\left(\cos(\tfrac{1}{3}\pi) + i\sin(\tfrac{1}{3}\pi)\right) \;\vee\; z = \sqrt[3]{8}\left(\cos(\pi) + i\sin(\pi)\right) \;\vee\; z = \sqrt[3]{8}\left(\cos(\tfrac{5}{3}\pi) + i\sin(\tfrac{5}{3}\pi)\right)$

$z = 2\left(\tfrac{1}{2} + \tfrac{1}{2}i\sqrt{3}\right) = 1 + i\sqrt{3} \;\vee\; z = 2(-1 + 0 \cdot i) = -2 \;\vee\; z = 2\left(\tfrac{1}{2} - \tfrac{1}{2}i\sqrt{3}\right) = 1 - i\sqrt{3}$.

17a $z^2 = -4i = 4e^{-\frac{1}{2}\pi i + k \cdot 2\pi i}$

$z = 2e^{-\frac{1}{4}\pi i + k \cdot \pi i}$

$z = 2e^{-\frac{1}{4}\pi i} \;\vee\; z = 2e^{\frac{3}{4}\pi i}$.

17b $z^2 = 9i = 9e^{\frac{1}{2}\pi i + k \cdot 2\pi i}$

$z = 3e^{\frac{1}{4}\pi i + k \cdot \pi i}$

$z = 3e^{\frac{1}{4}\pi i} \;\vee\; z = 3e^{1\frac{1}{4}\pi i} = 3e^{-\frac{3}{4}\pi i}$.

17c $z^3 = 27 = 27e^{k \cdot 2\pi i}$

$z = 3e^{k \cdot \frac{2}{3}\pi i}$

$z = 3e^0 = 3 \;\vee\; z = 3e^{\frac{2}{3}\pi i} \;\vee\; z = 3e^{\frac{4}{3}\pi i} = 3e^{-\frac{2}{3}\pi i}$.

17d $z^4 = -81 = 81e^{\pi i + k \cdot 2\pi i}$

$z = 3e^{\frac{1}{4}\pi i + k \cdot \frac{1}{2}\pi i}$

$z = 3e^{\frac{1}{4}\pi i} \;\vee\; z = 3e^{\frac{3}{4}\pi i} \;\vee\; z = 3e^{1\frac{1}{4}\pi i} \;\vee\; z = 3e^{1\frac{3}{4}\pi i}$.

18a $(z-1)^2 = \tfrac{1}{2} + \tfrac{1}{2}i\sqrt{3} = e^{\frac{1}{3}\pi i + k \cdot 2\pi i}$

$z - 1 = e^{\frac{1}{6}\pi i + k \cdot \pi i} \Rightarrow z = 1 + e^{\frac{1}{6}\pi i + k \cdot \pi i}$

$z = 1 + e^{\frac{1}{6}\pi i} \;\vee\; z = 1 + e^{1\frac{1}{6}\pi i}$

$z = 1 + \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i \;\vee\; z = 1 - \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}i$.

18b $(z + 4i)^4 = -16 = 16e^{\pi i + k \cdot 2\pi i}$

$z + 4i = 2e^{\frac{1}{4}\pi i + k \cdot \frac{1}{2}\pi i}$

$z = -4i + 2e^{\frac{1}{4}\pi i + k \cdot \frac{1}{2}\pi i}$

$z = -4i + 2e^{\frac{1}{4}\pi i} \;\vee\; z = -4i + 2e^{\frac{3}{4}\pi i} \;\vee\; z = -4i + 2e^{1\frac{1}{4}\pi i} \;\vee\; z = -4i + 2e^{1\frac{3}{4}\pi i}$

$z = -4i + 2\left(\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}i\sqrt{2}\right) \;\vee\; z = -4i + 2\left(-\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}i\sqrt{2}\right) \;\vee\; z = -4i + 2\left(-\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}i\sqrt{2}\right) \;\vee\; z = -4i + 2\left(\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}i\sqrt{2}\right)$

$z = \sqrt{2} + i\left(-4 + \sqrt{2}\right) \;\vee\; z = -\sqrt{2} + i\left(-4 + \sqrt{2}\right) \;\vee\; z = -\sqrt{2} + i\left(-4 - \sqrt{2}\right) \;\vee\; z = \sqrt{2} + i\left(-4 - \sqrt{2}\right)$.

18c $(z - 5i)^2 = \tfrac{1}{2} - \tfrac{1}{2}i\sqrt{3} = e^{-\frac{1}{3}\pi i + k \cdot 2\pi i}$

$z - 5i = e^{-\frac{1}{6}\pi i + k \cdot \pi i}$

$z = 5i + e^{-\frac{1}{6}\pi i + k \cdot \pi i}$

$z = 5i + e^{-\frac{1}{6}\pi i} \;\vee\; z = 5i + e^{\frac{5}{6}\pi i}$

$z = 5i + \tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}i \;\vee\; z = 5i - \tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}i$

$z = \tfrac{1}{2}\sqrt{3} + 4\tfrac{1}{2}i \;\vee\; z = -\tfrac{1}{2}\sqrt{3} + 5\tfrac{1}{2}i$.

18d $z^2 - 6z + 10 = (z-3)^2 - 9 + 10 = i\sqrt{3}$

$(z-3)^2 = -1 + i\sqrt{3} = 2e^{\frac{2}{3}\pi i + k \cdot 2\pi i}$

$z - 3 = \sqrt{2} \cdot e^{\frac{1}{3}\pi i + k \cdot \pi i} \Rightarrow z = 3 + \sqrt{2} \cdot e^{\frac{1}{3}\pi i + k \cdot \pi i}$

$z = 3 + \sqrt{2} \cdot e^{\frac{1}{3}\pi i} \;\vee\; z = 3 + \sqrt{2} \cdot e^{1\frac{1}{3}\pi i}$

$z = 3 + \sqrt{2} \cdot \left(\tfrac{1}{2} + \tfrac{1}{2}i\sqrt{3}\right) \;\vee\; z = 3 + \sqrt{2} \cdot \left(-\tfrac{1}{2} - \tfrac{1}{2}i\sqrt{3}\right)$

$z = 3 + \tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}i\sqrt{6} \;\vee\; z = 3 - \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}i\sqrt{6}$.

19a $z^2 = -25i = 25e^{-\frac{1}{2}\pi i + k\cdot 2\pi i}$

$z = 5e^{-\frac{1}{4}\pi i + k\cdot \pi i}$

$z = 5e^{-\frac{1}{4}\pi i} \vee z = 5e^{\frac{3}{4}\pi i}.$

19b $(2z-i)^2 = 25$

$2z - i = 5 \vee 2z - i = -5$

$2z = 5 + i \vee 2z = -5 + i$

$z = 2\frac{1}{2} + \frac{1}{2}i \vee z = -2\frac{1}{2} + \frac{1}{2}i.$

19c $z^2 - 8z = -16 + 9i$

$z^2 - 8z + 16 = 9i$

$(z-4)^2 = 9i = 9e^{\frac{1}{2}\pi i + k\cdot 2\pi i}$

$z - 4 = 3e^{\frac{1}{4}\pi i + k\cdot \pi i}$

$z = 4 + 3e^{\frac{1}{4}\pi i + k\cdot \pi i}$

$z = 4 + 3e^{\frac{1}{4}\pi i} \vee z = 4 + 3e^{1\frac{1}{4}\pi i}$

$z = 4 + 3\left(\frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}\right) \vee z = 4 + 3\left(-\frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}\right)$

$z = 4 + 1\frac{1}{2}\sqrt{2} + 1\frac{1}{2}i\sqrt{2} \vee z = 4 - 1\frac{1}{2}\sqrt{2} - 1\frac{1}{2}i\sqrt{2}.$

19d $(4z-i)^4 = 2\sqrt{2} + 2i\sqrt{2} = 4e^{\frac{1}{4}\pi i + k\cdot 2\pi i}$

$4z - i = \sqrt{2}\cdot e^{\frac{1}{16}\pi i + k\cdot \frac{1}{2}\pi i}$

$4z = i + \sqrt{2}\cdot e^{\frac{1}{16}\pi i + k\cdot \frac{1}{2}\pi i}$

$z = \frac{1}{4}i + \frac{1}{4}\sqrt{2}\cdot e^{\frac{1}{16}\pi i + k\cdot \frac{1}{2}\pi i}$

$z = \frac{1}{4}i + \frac{1}{4}\sqrt{2}\cdot e^{\frac{1}{16}\pi i} \vee z = \frac{1}{4}i + \frac{1}{4}\sqrt{2}\cdot e^{\frac{9}{16}\pi i} \vee z = \frac{1}{4}i + \frac{1}{4}\sqrt{2}\cdot e^{1\frac{1}{16}\pi i} \vee z = \frac{1}{4}i + \frac{1}{4}\sqrt{2}\cdot e^{1\frac{9}{16}\pi i}$

$z = \frac{1}{4}\sqrt{2}\cdot\cos(\frac{1}{16}\pi) + \left(\frac{1}{4} + \frac{1}{4}\sqrt{2}\cdot\sin(\frac{1}{16}\pi)\right)i \vee z = \frac{1}{4}\sqrt{2}\cdot\cos(\frac{9}{16}\pi) + \left(\frac{1}{4} + \frac{1}{4}\sqrt{2}\cdot\sin(\frac{9}{16}\pi)\right)i \vee$

$z = \frac{1}{4}\sqrt{2}\cdot\cos(1\frac{1}{16}\pi) + \left(\frac{1}{4} + \frac{1}{4}\sqrt{2}\cdot\sin(1\frac{1}{16}\pi)\right)i \vee z = \frac{1}{4}\sqrt{2}\cdot\cos(1\frac{9}{16}\pi) + \left(\frac{1}{4} + \frac{1}{4}\sqrt{2}\cdot\sin(1\frac{9}{16}\pi)\right)i.$

20a $f(0) = e^0 = 1;$ $f(1) = e^1 = e$ en $f(-2) = e^{-2} = \frac{1}{e^2}.$

20b De functie $f(z) = e^z$ geeft voor elk reëel origineel een positief reëel beeld, dus de functie $f(z) = e^z$ beeldt de reële as af op de positieve reële as.

20c $\left|f(\frac{1}{4}\pi i)\right| = \left|e^{\frac{1}{4}\pi i}\right| = \left|\cos(\frac{1}{4}\pi) + i\sin(\frac{1}{4}\pi)\right| = 1$ en $\text{Arg}\left(f(\frac{1}{4}\pi)\right) = \text{Arg}(e^{\frac{1}{4}\pi i}) = \text{Arg}\left(\cos(\frac{1}{4}\pi) + i\sin(\frac{1}{4}\pi)\right) = \frac{1}{4}\pi.$

20d $\left|f(\frac{3}{4}\pi i)\right| = \left|e^{\frac{3}{4}\pi i}\right| = \left|\cos(\frac{3}{4}\pi) + i\sin(\frac{3}{4}\pi)\right| = 1$ en $\text{Arg}\left(f(\frac{3}{4}\pi)\right) = \text{Arg}(e^{\frac{3}{4}\pi i}) = \text{Arg}\left(\cos(\frac{3}{4}\pi) + i\sin(\frac{3}{4}\pi)\right) = \frac{3}{4}\pi.$

20e $\left|f(2\frac{1}{4}\pi i)\right| = \left|e^{2\frac{1}{4}\pi i}\right| = \left|\cos(2\frac{1}{4}\pi) + i\sin(2\frac{1}{4}\pi)\right| = 1$ en $\text{Arg}\left(f(2\frac{1}{4}\pi i)\right) = \text{Arg}(e^{2\frac{1}{4}\pi i}) = \text{Arg}\left(\cos(2\frac{1}{4}\pi) + i\sin(2\frac{1}{4}\pi)\right) = \frac{1}{4}\pi.$

20f $\left|f(\alpha i)\right| = \left|e^{\alpha i}\right| = \left|\cos(\alpha i) + i\sin(\alpha i)\right| = 1$ en $\arg(e^{\alpha i}) = \arg(\cos(\alpha i) + i\sin(\alpha i)) = \alpha.$

Dus de functie $f(z) = e^z$ beeldt de imaginaire as af op de eenheidscirkel.

20g $f(z + k\cdot 2\pi i) = e^{z + k\cdot 2\pi i} = e^z\cdot e^{k\cdot 2\pi i} = e^z\cdot\left(e^{2\pi i}\right)^k = e^z\cdot\left(e^{0i}\right)^k = e^z\cdot\left(e^0\right)^k = e^z\cdot 1^k = e^z\cdot 1 = e^z.$

Dus de functie $f(z) = e^z$ is periodiek.

21a $\ln(e^{i\varphi}) = i\varphi$ geeft voor $\varphi = \pi$ dat $\ln(e^{\pi i}) = \pi i$, maar $e^{\pi i} = -1$, want $\left|e^{\pi i}\right| = 1$ en $\text{Arg}(e^{\pi i}) = \pi \Rightarrow \ln(e^{\pi i}) = \ln(-1) = \pi i.$

21b $-1 = e^{\pi i} = e^{3\pi i} \Rightarrow \ln(-1) = \ln(e^{3\pi i}) = 3\pi i.$

21c $f(i) = \ln(i) = \ln(e^{\frac{1}{2}\pi i}) = \frac{1}{2}\pi i.$

22a Noem $z = a + bi$ dan $f(z) = f(a + bi) = e^{a + bi} = e^a\cdot e^{bi} = e^a\left(\cos(b) + i\sin(b)\right)$, met $\left|f(z)\right| = \left|e^z\right| = e^a$ en $\text{Arg}(f(z)) = \text{Arg}(e^z) = b.$

Bij vaste a en variabele b krijg je dus de punten in het complexe getallen die op afstand e^a van $z = 0$ af liggen en waarbij de draaiingshoek b kan voorkomen.

Dus bij de functie $f(z) = e^z$ is het beeld van $z = a + ib$ met a vast de cirkel met middelpunt $z = 0$ en straal e^a.

22b Bij vaste a en vaste b is $e^a\left(\cos(b) + i\sin(b)\right)$ een punt in het complexe vlak met afstand e^a van $z = 0$ draaiingshoek b (rond $z = 0$) ten opzichte van de positieve reële as.

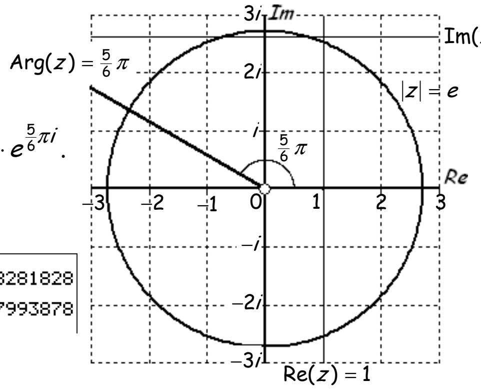
Bij vaste b en variabele a krijg je dus de punten in het complexe getallen die alle hetzelfde argument hebben, maar waarvan de afstanden e^a tot $z = 0$ verschillen.

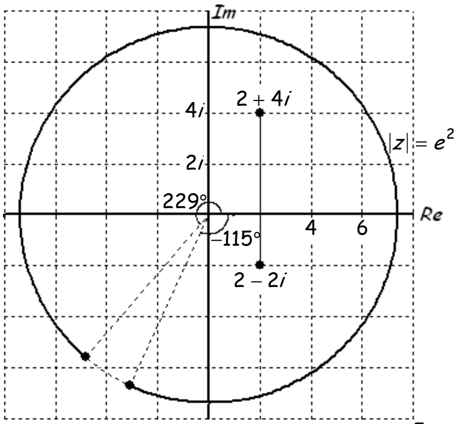
Deze punten vormen samen de halve lijn die een hoek van b radialen maakt met de positieve reële as.

Dus bij de functie $f(z) = e^z$ is het beeld van $z = a + ib$ met b vast de halve lijn die een hoek van b radialen maakt met de positieve reële as.

23 $\text{Re}(z) = 1$, dus $z = 1 + bi \Rightarrow f(z) = e^z = e^{1+bi} = e^1 \cdot e^{bi}$.
Het beeld van $\text{Re}(z) = 1$is de cirkel met middelpunt $z = 0$ en straal e^1 ($\approx 2{,}7$), ofwel de cirkel met de vergelijking $|z| = e$.
$\text{Im}(z) = \frac{5}{6}\pi$ ($\approx 2{,}6$), dus $z = a + \frac{5}{6}\pi i \Rightarrow f(z) = e^z = e^{a+\frac{5}{6}\pi i} = e^a \cdot e^{\frac{5}{6}\pi i}$.
Het beeld van $\text{Im}(z) = \frac{5}{6}\pi$ is de halve lijn vanaf $z = 0$ die een hoek van $\frac{5}{6}\pi$ radialen maakt met de positieve reële as, ofwel de halve lijn met vergelijking $\text{Arg}(z) = \frac{5}{6}\pi$.

```
e
            2.718281828
5/6π
            2.617993878
```



24a


Het beeld van $\text{Re}(z) = 2$ bij $f(z) = e^z$ is de cirkel met middelpunt $z = 0$ en straal e^2 ($\approx 7{,}4$).
Het beeld van $\text{Im}(z) = -2$ bij $f(z) = e^z$ heeft argument -2 radialen ($\approx -115°$) en van $\text{Im}(z) = 4$ heeft argument 4 radialen ($\approx 229°$).
Dus het beeld van het lijnstuk bestaat uit de complexe getallen z waarvoor $|z| = e^2$ en $-2 \le \arg(z) \le 4$.

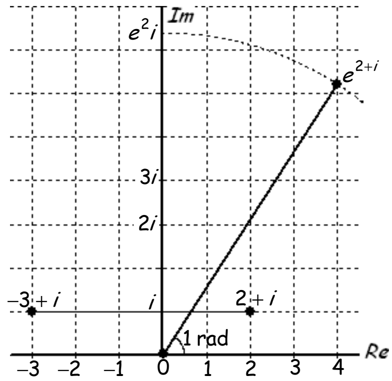
```
e²
             7.389056099
-2/π*180
            -114.591559
4/π*180
             229.1831181
```

```
e^(2-2i)
-3.074932321-6.…
e^(2+4i)
-4.829809383-5.…
```

24b Het beeld van $\text{Im}(z) = 1$ bij $f(z) = e^z$ bestaat uit de complexe getallen met argument 1 radiaal ($\approx -57°$).
Het beeld van $\text{Re}(z) = -3$ bij $f(z) = e^z$ heeft modulus e^{-3} ($\approx 0{,}05$) en het beeld van $\text{Re}(z) = 2$ heeft modulus e^2 ($\approx 7{,}4$)
Dus het beeld van het lijnstuk bestaat uit de complexe getallen z waarvoor $e^{-3} \le |z| \le e^2$ en $\text{Arg}(z) = 1$.

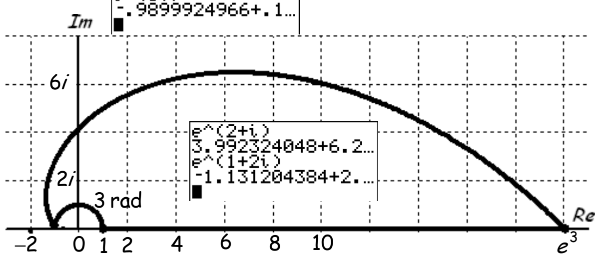
```
1/π*180
             57.29577951
e^(-3)
            .0497870684
e^(2)
             7.389056099
```

```
e^(-3+i)
.0269000678+.04…
e^(2+i)
3.992324048+6.2…
```



24c $f(0) = e^0 = 1$; $f(3) = e^3$ en $f(3i) = e^{3i}$.
Het beeld van het lijnstuk met eindpunten $z = 0$ en $z = 3$ is dat deel van de reële as tussen $z = 1$ en $z = e^3$.
Het beeld van het lijnstuk met eindpunten $z = 0$ en $z = 3i$ is dat deel van de cirkel met middelpunt $z = 0$ en straal 1 dat bestaat uit de getallen z waarvoor $0 \le \text{Arg}(z) \le 3$.
Het beeld van het lijnstuk met eindpunten $z = 3$ en $z = 3i$ is dat de kromme met beginpunt $z = e^3$ en die verder gaat door $f(2+i) \approx 3{,}99 + 6{,}22i$ en $f(1+2i) \approx -1{,}13 + 2{,}47i$ en eindpunt $f(3i) \approx -0{,}99 + 0{,}14i$.

```
e^(3)
             20.08553692
3/π*180
             171.8873385
e^(3i)
-.9899924966+.1…
```




```
e^(2+i)
3.992324048+6.2…
e^(1+2i)
-1.131204384+2.…
```

25a $f(-e) = \ln(-e) = \ln(-1 \cdot e) = \ln(-1) + \ln(e) = \ln(e^{\pi i}) + 1 = 1 + \pi i$ of $\ln(-1 \cdot e) = \ln(e^{\pi i} \cdot e^1) = \ln(e^{\pi i + 1}) = 1 + \pi i$.

25b $f(-e^2) = \ln(-e^2) = \ln(-1 \cdot e^2) = \ln(e^{\pi i} \cdot e^2) = \ln(e^{\pi i + 2}) = 2 + \pi i$.

25c $f(ei) = \ln(ei) = \ln(e \cdot i) = \ln(e^1 \cdot e^{\frac{1}{2}\pi i}) = \ln(e^{1 + \frac{1}{2}\pi i}) = 1 + \frac{1}{2}\pi i$.

25d $f(3i) = \ln(3i) = \ln(3 \cdot i) = \ln(3) + \ln(i) = \ln(3) + \ln(e^{\frac{1}{2}\pi i}) = \ln(3) + \frac{1}{2}\pi i$.

25e $f(-3) = \ln(-3) = \ln(-1 \cdot 3) = \ln(-1) + \ln(3) = \ln(e^{\pi i}) + \ln(3) = \ln(3) + \pi i$.

25f $f(-2i) = \ln(-2i) = \ln(2 \cdot -i) = \ln(2) + \ln(-i) = \ln(2) + \ln(e^{-\frac{1}{2}\pi i}) = \ln(2) - \frac{1}{2}\pi i$.

25g $f(2-2i) = \ln(2-2i) = \ln(2\sqrt{2} \cdot e^{-\frac{1}{4}\pi i}) = \ln(2\sqrt{2}) + \ln(e^{-\frac{1}{4}\pi i}) = \ln(2^{1\frac{1}{2}}) + \ln(e^{-\frac{1}{4}\pi i}) = 1\frac{1}{2}\ln(2) - \frac{1}{4}\pi i$.

25h $f(\sqrt{3} - i) = \ln(\sqrt{3} - i) = \ln(2 \cdot e^{-\frac{1}{6}\pi i}) = \ln(2) + \ln(e^{-\frac{1}{6}\pi i}) = \ln(2) - \frac{1}{6}\pi i$.

25i $f(e + ei) = \ln(e + ei) = \ln(e(1+i)) = \ln(e \cdot \sqrt{2}e^{\frac{1}{4}\pi i}) = \ln(e) + \ln(2^{\frac{1}{2}}) + \ln(e^{\frac{1}{4}\pi i}) = 1 + \frac{1}{2}\ln(2) + \frac{1}{4}\pi i$.

26a $f(-e\sqrt{e}) = \ln(-e^{1\frac12}) = \ln(-1\cdot e^{1\frac12}) = \ln(e^{\pi i}\cdot e^{1\frac12}) = \ln(e^{\pi i + 1\frac12}) = 1\frac12 + \pi i.$

26b $f(-i^2\sqrt{i}) = \ln(-i^2\sqrt{i}) = \ln(--1\sqrt{i}) = \ln(\sqrt{i}) = \ln(i^{\frac12}) = \frac12\ln(i) = \frac12\ln(e^{\frac12\pi i}) = \frac12\cdot\frac12\pi i = \frac14\pi i.$

26c $f\left(-\frac1e\right) = \ln\left(-\frac1e\right) = \ln(-1\cdot e^{-1}) = \ln(e^{\pi i}\cdot e^{-1}) = \ln(e^{\pi i-1}) = -1+\pi i.$

26d $f\left(\frac{2}{i\sqrt{i}}\right) = \ln\left(\frac{2}{i\sqrt{i}}\right) = \ln(2)-\ln(i\sqrt{i}) = \ln(2)-\ln(i^{\frac32}) = \ln(2)-\frac32\ln(i) = \ln(2)-\frac32\ln(e^{\frac12\pi i}) = \ln(2)-\frac32\cdot\frac12\pi i = \ln(2)-\frac34\pi i.$

26e $f\left(\frac{2}{1+i}\right)=\ln\left(\frac{2}{1+i}\right)=\ln(2)-\ln(1+i)=\ln(2)-\ln(\sqrt2\cdot e^{\frac14\pi i})=\ln(2)-\ln(2^{\frac12})-\ln(e^{\frac14\pi i})=\ln(2)-\frac12\ln(2)-\frac14\pi i=\frac12\ln(2)-\frac14\pi i.$

26f $f\left(\frac{e}{1+i\sqrt3}\right)=\ln\left(\frac{e}{1+i\sqrt3}\right)=\ln(e)-\ln(1+i\sqrt3)=1-\ln(2\cdot e^{\frac13\pi i})=1-(\ln(2)+\ln(e^{\frac13\pi i}))=1-\ln(2)-\ln(e^{\frac13\pi i})=1-\ln(2)-\frac13\pi i.$

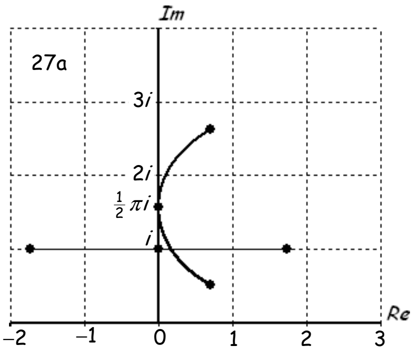
27a $f(-\sqrt3+i)=\ln(-\sqrt3+i)=\ln(2\cdot e^{\frac56\pi i})=\ln(2)+\ln(e^{\frac56\pi i})=\ln(2)+\frac56\pi i.$
$f(\sqrt3+i)=\ln(\sqrt3+i)=\ln(2\cdot e^{\frac16\pi i})=\ln(2)+\ln(e^{\frac16\pi i})=\ln(2)+\frac16\pi i.$
$f(i)=\ln(i)=\ln(e^{\frac12\pi i})=\frac12\pi i.$ (zie de eerste figuur hieronder)

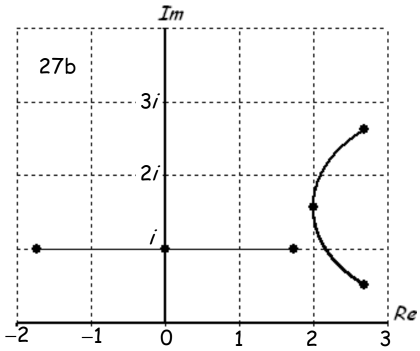
```
ln(2)
      .6931471806
5/6π  2.617993878
1/6π
      .5235987756
1/2π  1.570796327
```

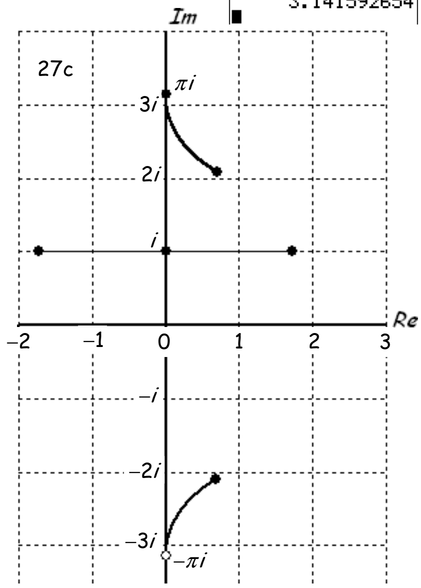
27b $g\left(e^2\cdot(-\sqrt3+i)\right)=\ln\left(e^2\cdot(-\sqrt3+i)\right)=\ln(e^2\cdot 2e^{\frac56\pi i})=\ln(e^2)+\ln(2)+\ln(e^{\frac56\pi i})=2+\ln(2)+\frac56\pi i.$
$g\left(e^2\cdot(\sqrt3+i)\right)=\ln\left(e^2\cdot(\sqrt3+i)\right)=\ln(e^2\cdot 2e^{\frac16\pi i})=\ln(e^2)+\ln(2)+\ln(e^{\frac16\pi i})=2+\ln(2)+\frac16\pi i.$
$g(e^2 i)=\ln(e^2\cdot i)=\ln(e^2\cdot e^{\frac12\pi i})=\ln(e^2)+\ln(e^{\frac12\pi i})=2+\frac12\pi i.$

27c $h\left(i\cdot(-\sqrt3+i)\right)=\ln\left(i\cdot(-\sqrt3+i)\right)=\ln(e^{\frac12\pi i}\cdot 2e^{\frac56\pi i})=\ln(2)+\ln(e^{\frac43\pi i})=\ln(2)+\ln(e^{-\frac23\pi i})=\ln(2)-\frac23\pi i.$
In dit hoofdstuk beperken we ons tot $-\pi<\mathrm{Im}(f(z))\le\pi$. (zie bovenaan blz. 145)
$h\left(i\cdot(\sqrt3+i)\right)=\ln\left(i\cdot(\sqrt3+i)\right)=\ln(e^{\frac12\pi i}\cdot 2e^{\frac16\pi i})=\ln(2)+\ln(e^{\frac23\pi i})=\ln(2)+\frac23\pi i.$
$h(i\cdot i)=\ln(i^2)=2\ln(i)=2\ln(e^{\frac12\pi i})=2\cdot\frac12\pi i=\pi i.$

```
-2/3π -2.094395102
2/3π   2.094395102
π      3.141592654
```







28a $i^i=(e^{\frac12\pi i})^i=e^{\frac12\pi i^2}=e^{-\frac12\pi}.$

28b $i^{2i}=(e^{\frac12\pi i})^{2i}=e^{\pi i^2}=e^{-\pi}$ en
$i^{1+i}=i^1\cdot i^i=i\cdot(e^{\frac12\pi i})^i=i\cdot e^{\frac12\pi i^2}=i\cdot e^{-\frac12\pi}.$

29 $\cos(x)=\frac{e^{ix}+e^{-ix}}{2}\Rightarrow\cos(i)=\frac{e^{i^2}+e^{-i^2}}{2}=\frac{e^{-1}+e^{1}}{2}=\frac{e^{-1}}{2}+\frac{e^1}{2}=\frac12\cdot e^{-1}+\frac e2=\frac12\cdot\frac1e+\frac e2=\frac{1}{2e}+\frac e2.$

30a $\cos(\frac16\pi+i)=\frac{e^{i(\frac16\pi+i)}+e^{-i(\frac16\pi+i)}}{2}=\frac{e^{\frac16\pi i-1}+e^{-\frac16\pi i+1}}{2}=\frac{e^{-1}\cdot e^{\frac16\pi i}+e\cdot e^{-\frac16\pi}}{2}=\frac1e\cdot\frac{\frac12\sqrt3+\frac12 i}{2}+\frac{e\cdot(\frac12\sqrt3-\frac12 i)}{2}=\frac{\sqrt3}{4e}+\frac{e\sqrt3}{4}+\left(\frac{1}{4e}-\frac e4\right)i.$

30b $\sin(i)=\frac{e^{i\cdot i}-e^{-i\cdot i}}{2i}=\frac{e^{-1}-e}{2i}=\frac1e\cdot\frac{1}{2i}-\frac{e}{2i}=\frac{1}{2ei}\cdot\frac{-i}{-i}-\frac{e}{2i}\cdot\frac{-i}{-i}=\frac{-i}{2e}-\frac{-ei}{2}=-\frac{1}{2e}i+\frac e2 i=\left(\frac e2-\frac{1}{2e}\right)i.$

30c $\cos(\frac13\pi+2i)=\frac{e^{i(\frac13\pi+2i)}+e^{-i(\frac13\pi+2i)}}{2}=\frac{e^{\frac13\pi i-2}+e^{-\frac13\pi i+2}}{2}=\frac{e^{-2}\cdot e^{\frac13\pi i}+e^2\cdot e^{-\frac13\pi i}}{2}=\frac{\frac12+\frac12 i\sqrt3}{2e^2}+\frac{e^2(\frac12-\frac12 i\sqrt3)}{2}=\frac{1}{4e^2}+\frac{e^2}{4}+\left(\frac{\sqrt3}{4e^2}-\frac{e^2\sqrt3}{4}\right)i.$

30d $\sin(\frac14\pi-3i)=\frac{e^{i(\frac14\pi-3i)}-e^{-i(\frac14\pi-3i)}}{2i}=\frac{e^{\frac14\pi i+3}-e^{-\frac13\pi-3}}{2i}=\frac{e^3\cdot e^{\frac14\pi i}-e^{-3}\cdot e^{-\frac14\pi}}{2i}=\frac{e^3(\frac12\sqrt2+\frac12 i\sqrt2)-e^{-3}(\frac12\sqrt2-\frac12 i\sqrt2)}{2i}$
$=\frac{e^3(\frac12\sqrt2+\frac12 i\sqrt2)-e^{-3}(\frac12\sqrt2-\frac12 i\sqrt2)}{2i}\cdot\frac{-i}{-i}=\frac{e^3(-\frac12 i\sqrt2+\frac12\sqrt2)-e^{-3}(-\frac12 i\sqrt2-\frac12\sqrt2)}{2}=\frac{e^3\sqrt2}{4}+\frac{\sqrt2}{4e^3}+\left(\frac{\sqrt2}{4e^3}-\frac{e^3\sqrt2}{4}\right)i.$

31a $\cos^2(i)+\sin^2(i)=\left(\frac{e^{i\cdot i}+e^{-i\cdot i}}{2}\right)^2+\left(\frac{e^{i\cdot i}-e^{-i\cdot i}}{2i}\right)^2$

$=\left(\frac{e^{-1}+e}{2}\right)^2+\left(\frac{e^{-1}-e}{2i}\right)^2$

$=\frac{e^{-2}+2+e^2}{4}+\frac{e^{-2}-2+e^2}{-4}$

$=\frac{e^{-2}+2+e^2-e^{-2}+2-e^2}{4}=\frac{4}{4}=1.$

31b $\sin(2i)=2\sin(i)\cos(i)$

$\frac{e^{i\cdot 2i}-e^{-i\cdot 2i}}{2i}=2\cdot\frac{e^{i\cdot i}-e^{-i\cdot i}}{2i}\cdot\frac{e^{i\cdot i}+e^{-i\cdot i}}{2}$

$\frac{e^{-2}-e^2}{2i}=2\cdot\frac{e^{-1}-e}{2i}\cdot\frac{e^{-1}+e}{2}$

$\frac{e^{-2}-e^2}{2i}=2\cdot\frac{e^{-2}-e^2}{4i}$

$\frac{e^{-2}-e^2}{2i}=\frac{e^{-2}-e^2}{2i}.$

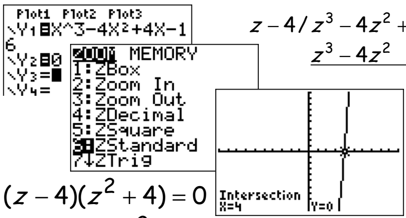
32a $1^3+1^2+1-3=1+1+1-3=0$. Dus $x=1$ is een oplossing van de vergelijking $x^3+x^2+x-3=0$.

32b $(-1+i\sqrt{2})^3+(-1+i\sqrt{2})^2+(-1+i\sqrt{2})-3=(-1+i\sqrt{2})(-1+i\sqrt{2})^2+(-1+i\sqrt{2})^2-1+i\sqrt{2}-3$

$=(-1+i\sqrt{2})(1-2i\sqrt{2}-2)+1-2i\sqrt{2}-2-1+i\sqrt{2}-3$

$=-1+2i\sqrt{2}+2+i\sqrt{2}+4-2i\sqrt{2}+1-2i\sqrt{2}-2-1+i\sqrt{2}-3=-1+2+4+1-2-1-3=0$. Klopt.

33a $z^3-4z^2+4z-16=0$ (intersect) $\Rightarrow z=4$.



$z-4\,/\,z^3-4z^2+4z-16\,\backslash\,z^2+4$
$\underline{z^3-4z^2}\;-$
$4z-16$
$\underline{4z-16}\;-$
0

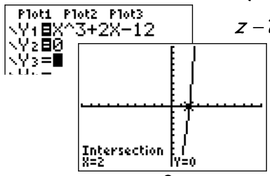
$(z-4)(z^2+4)=0$

$z=4 \;\vee\; z^2=-4$

$z=4 \;\vee\; z^2=4i^2$

$z=4 \;\vee\; z=-2i \;\vee\; z=2i.$

33b $z^3+2z-12=0$ (intersect) $\Rightarrow z=2$.



$z-2\,/\,z^3\quad+2z-12\,\backslash\,z^2+2z+6$
$\underline{z^3-2z^2}\;-$
$2z^2+2z-12$
$\underline{2z^2-4z}\;-$
$6z-12$
$\underline{6z-12}\;-$
0

$(z-2)(z^2+2z+6)=0$

$z=2 \;\vee\; z^2+2z+6=0$

$z=2 \;\vee\; (z+1)^2-1+6=0$

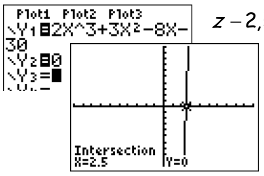
$z=2 \;\vee\; (z+1)^2=-5$

$z=2 \;\vee\; (z+1)^2=5i^2$

$z=2 \;\vee\; z+1=-i\sqrt{5} \;\vee\; z+1=i\sqrt{5}$

$z=2 \;\vee\; z=-1-i\sqrt{5} \;\vee\; z=-1+i\sqrt{5}.$

33c $2z^3+3z^2-8z-30=0$ (intersect) $\Rightarrow z=2\frac{1}{2}$.



$z-2{,}5\,/\,2z^3+3z^2-8z-30\,\backslash\,2z^2+8z+12$
$\underline{2z^3-5z^2}\;-$
$8z^2-8z-30$
$\underline{8z^2-20z}\;-$
$12z-30$
$\underline{12z-30}\;-$
0

$(z-2\frac{1}{2})(2z^2+8z+12)=0$

$z=2\frac{1}{2} \;\vee\; 2z^2+8z+12=0$

$z=2\frac{1}{2} \;\vee\; z^2+4z+6=0$

$z=2\frac{1}{2} \;\vee\; (z+2)^2-4+6=0$

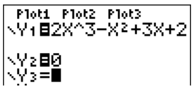
$z=2\frac{1}{2} \;\vee\; (z+2)^2=-2$

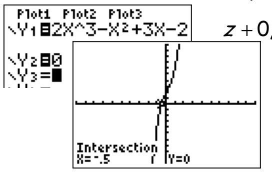
$z=2\frac{1}{2} \;\vee\; (z+2)^2=2i^2$

$z=2\frac{1}{2} \;\vee\; z+2=-i\sqrt{2} \;\vee\; z+2=i\sqrt{2}$

$z=2\frac{1}{2} \;\vee\; z=-2-i\sqrt{2} \;\vee\; z=-2+i\sqrt{2}.$

33d $2z^3-z^2+3z+2=0$ (intersect) $\Rightarrow z=-\frac{1}{2}$.





$z+0{,}5\,/\,2z^3-z^2+3z+2\,\backslash\,2z^2-2z+4$
$\underline{2z^3+z^2}\;-$
$-2z^2+3z+2$
$\underline{-2z^2-\;z}\;-$
$4z+2$
$\underline{4z+2}\;-$
0

$(z+\frac{1}{2})(2z^2-2z+4)=0$

$z=-\frac{1}{2} \;\vee\; z^2-z+2=0$

$z=-\frac{1}{2} \;\vee\; (z-\frac{1}{2})^2-\frac{1}{4}+2=0$

$z=-\frac{1}{2} \;\vee\; (z-\frac{1}{2})^2=-1\frac{3}{4}$

$z=-\frac{1}{2} \;\vee\; (z-\frac{1}{2})^2=\frac{7}{4}i^2$

$z=-\frac{1}{2} \;\vee\; z-\frac{1}{2}=-\frac{1}{2}i\sqrt{7} \;\vee\; z-\frac{1}{2}=\frac{1}{2}i\sqrt{7}$

$z=-\frac{1}{2} \;\vee\; z=\frac{1}{2}-\frac{1}{2}i\sqrt{7} \;\vee\; z=\frac{1}{2}+\frac{1}{2}i\sqrt{7}.$

34a $(u+v)^3+6(u+v)=20$

$(u+v)(u+v)^2+6u+6v=20$

$(u+v)(u^2+2uv+v^2)+6u+6v=20$

$u^3+2u^2v+uv^2+u^2v+2uv^2+v^3+6u+6v=20$

Dus $u^3+3u^2v+3uv^2+v^3+6u+6v=20$ **(1)**

34b $u^3+3u\cdot uv+3v\cdot uv+v^3+6u+6v=20$

$u^3+3u\cdot -2+3v\cdot -2+v^3+6u+6v=20$

$u^3-6u-6v+v^3+6u+6v=20$

$u^3+v^3=20$ **(2)**

34c $u^3+\left(-\frac{2}{u}\right)^3=20$

$u^3-\frac{2^3}{u^3}=20$

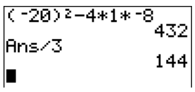
$u^3-\frac{8}{u^3}=20$ (vermenigvuldigen met u^3)

$u^6-8=20u^3$

$u^6-20u^3-8=0$ **(3)**

34d Stel (in 34c) $u^3=x$. Dit geeft $x^2-20x-8=0$

$D=(-20)^2-4\cdot1\cdot-8=432\Rightarrow\sqrt{D}=\sqrt{432}.$



$x=\frac{20-\sqrt{432}}{2\cdot1} \;\vee\; x=\frac{20+\sqrt{432}}{2\cdot1}$. Dus $u^3=\frac{20-\sqrt{432}}{2} \;\vee\; u^3=\frac{20+\sqrt{432}}{2}$.

34e $\left.\begin{array}{l}u^3+v^3=20\Rightarrow v^3=20-u^3\\ u^3=10+6\sqrt{3}\end{array}\right\}\Rightarrow$

$v^3=20-\left(10+6\sqrt{3}\right)=10-6\sqrt{3}.$

34f $u^3 = 10 + 6\sqrt{3} \Rightarrow u = \sqrt[3]{10 + 6\sqrt{3}}$ en $v^3 = 10 - 6\sqrt{3} \Rightarrow v = \sqrt[3]{10 - 6\sqrt{3}}$. Dit geeft $u + v = \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}}$.

34g $\sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} = 2$.
Substitutie in $z^3 + 6z = 20$ geeft $2^3 + 6 \cdot 2 = 20 \Rightarrow 8 + 12 = 20 \Rightarrow 20 = 20$. Klopt!

```
3*√(10+6√(3))+3*
√(10-6√(3))
                2
```

35 $(u+v)^3 - 3uv(u+v) = q$ (haakjes wegwerken)
$u^3 + 3u^2v + 3uv^2 + v^3 - 3u^2v - 3uv^2 = q$ (vereenvoudigen)
$u^3 + v^3 = q$.

36 Verder rekenen in opgave 34 met $u^3 = 10 - 6\sqrt{3}$ geeft $v^3 = 20 - \left(10 - 6\sqrt{3}\right) = 10 + 6\sqrt{3}$.
Dit geeft $z = u + v = \sqrt[3]{10 - 6\sqrt{3}} + \sqrt[3]{10 + 6\sqrt{3}} = 2$ en dus hetzelfde resultaat.
Verder rekenen in het voorbeeld boven opgave 35 met $u^3 = -1$ geeft $v^3 = 63 - -1 = 64$.
Dit geeft $z = u + v = \sqrt[3]{-1} + \sqrt[3]{64} = -1 + 4 = 3$ en dus hetzelfde resultaat.

37a $z^3 + 36z = 208$
$\left.\begin{array}{l} z = u + v \text{ en } 36 = -3uv \text{ geeft } u^3 + v^3 = 208 \\ \text{Uit } -3uv = 36 \text{ volgt } v = -\frac{12}{u} \end{array}\right\} \Rightarrow$
$u^3 + \left(-\frac{12}{u}\right)^3 = 208$
$u^3 - \frac{1728}{u^3} = 208$
$u^6 - 208u^3 - 1728 = 0$

```
(-12)^3
            -1728
208²-4*1*-1728
            50176
√(Ans)
              224
```

$D = (-208)^2 - 4 \cdot 1 \cdot -1728 = 50176 \Rightarrow \sqrt{D} = 224$
$u^3 = \frac{208 + 224}{2} = 216$ is een oplossing
$v^3 = 208 - u^3 = 208 - 216 = -8$
$z = u + v = \sqrt[3]{216} + \sqrt[3]{-8} = 6 - 2 = 4$

```
3*√216+3*√-8
                4
```

Nu de staartdeling maken: $z - 4 / z^3 \quad + 36z - 208 \backslash z^2 + 4z + 52$
$z^3 - 4z^2$ −
$4z^2 + 36z - 208$
$4z^2 - 16z$ −
$52z - 208$
$52z - 208$ −
0

$z^3 + 36z - 208 = 0$
$(z-4)(z^2 + 4z + 52) = 0$
$z = 4 \ \vee \ z^2 + 4z + 52 = 0$
$z = 4 \ \vee \ (z+2)^2 - 4 + 52 = 0$
$z = 4 \ \vee \ (z+2)^2 = -48$
$z = 4 \ \vee \ (z+2)^2 = 48i^2$

```
48/3
               16
```

$z = 4 \ \vee \ z + 2 = 4i\sqrt{3} \ \vee \ z + 2 = -4i\sqrt{3}$
$\underline{\underline{z = 4 \ \vee \ z = -2 + 4i\sqrt{3} \ \vee \ z = -2 - 4i\sqrt{3}.}}$

37b $z^3 + 18z = 215$
$\left.\begin{array}{l} z = u + v \text{ en } 18 = -3uv \text{ geeft } u^3 + v^3 = 215 \\ \text{Uit } -3uv = 18 \text{ volgt } v = -\frac{6}{u} \end{array}\right\} \Rightarrow$
$u^3 + \left(-\frac{6}{u}\right)^3 = 215$

```
(-6)^3
             -216
```

$u^3 - \frac{216}{u^3} = 215$
$u^6 - 215u^3 - 216 = 0$
$(u^3 - 216)(u^3 + 1) = 0$
$u^3 = 216 \ \left(\vee \ u^3 = -1\right)$
$u^3 = 216$ geeft $v^3 = 215 - u^3 = 215 - 216 = -1$
$z = u + v = \sqrt[3]{216} + \sqrt[3]{-1} = 6 - 1 = 5$ (hiernaast verder)

De staartdeling: $z - 5 / z^3 \quad + 18z - 215 \backslash z^2 + 5z + 43$
$z^3 - 5z^2$ −
$5z^2 + 18z - 215$
$5z^2 - 25z$ −
$43z - 215$
$43z - 215$ −
0

$z^3 + 18z - 215 = 0$
$(z-5)(z^2 + 5z + 43) = 0$
$z = 5 \ \vee \ z^2 + 5z + 43 = 0$
$z = 5 \ \vee \ (z + \frac{5}{2})^2 - \frac{25}{4} + 43 = 0$
$z = 5 \ \vee \ (z + \frac{5}{2})^2 = -\frac{147}{4}$
$z = 5 \ \vee \ (z + \frac{5}{2})^2 = \frac{147}{4} i^2$

```
-43+25/4▸Frac
           -147/4
147/3
               49
```

$z = 5 \ \vee \ z + \frac{5}{2} = \frac{7}{2} i\sqrt{3} \ \vee \ z + \frac{5}{2} = -\frac{7}{2} i\sqrt{3}$
$\underline{\underline{z = 5 \ \vee \ z = -\frac{5}{2} + \frac{7}{2} i\sqrt{3} \ \vee \ z = -\frac{5}{2} - \frac{7}{2} i\sqrt{3}.}}$

37c $z^3 + 2\frac{1}{4}z = 3\frac{1}{4}$
$\left.\begin{array}{l} z = u + v \text{ en } 2\frac{1}{4} = -3uv \text{ geeft } u^3 + v^3 = 3\frac{1}{4} \\ \text{Uit } -3uv = 2\frac{1}{4} = \frac{9}{4} \text{ volgt } v = -\frac{3}{4u} \end{array}\right\} \Rightarrow$
$u^3 + \left(-\frac{3}{4u}\right)^3 = 3\frac{1}{4}$

```
3^3
               27
4^3
               64
```

$u^3 - \frac{27}{64u^3} = 3\frac{1}{4}$
$u^6 - 3\frac{1}{4}u^3 - \frac{27}{64} = 0$
$D = (-3\frac{1}{4})^2 - 4 \cdot 1 \cdot -\frac{27}{64} = \frac{49}{4} \Rightarrow \sqrt{D} = \frac{7}{2}$

```
3.25²-4*1*-27/64
            12.25
√(Ans)
              3.5
```

$u^3 = \frac{\frac{13}{4} + \frac{7}{2}}{2} = \frac{\frac{13}{4} + \frac{14}{4}}{2} = \frac{\frac{27}{4}}{2} = \frac{27}{8}$ is een oplossing
$v^3 = 3\frac{1}{4} - u^3 = \frac{13}{4} - \frac{27}{8} = \frac{26}{8} - \frac{27}{8} = -\frac{1}{8}$
$z = u + v = \sqrt[3]{\frac{27}{8}} + \sqrt[3]{-\frac{1}{8}} = \frac{3}{2} - \frac{1}{2} = 1$ (hiernaast verder)

De staartdeling: $z - 1 / z^3 \quad + 2\frac{1}{4}z - 3\frac{1}{4} \backslash z^2 + z + 3\frac{1}{4}$
$z^3 - z^2$ −
$z^2 + 2\frac{1}{4}z - 3\frac{1}{4}$
$z^2 - \ z$ −
$3\frac{1}{4}z - 3\frac{1}{4}$
$3\frac{1}{4}z - 3\frac{1}{4}$ −
0

$z^3 + 2\frac{1}{4}z - 3\frac{1}{4} = 0$
$(z-1)(z^2 + z + 3\frac{1}{4}) = 0$
$z = 1 \ \vee \ z^2 + z + 3\frac{1}{4} = 0$
$z = 1 \ \vee \ (z + \frac{1}{2})^2 - \frac{1}{4} + 3\frac{1}{4} = 0$
$z = 1 \ \vee \ (z + \frac{1}{2})^2 = -3$
$z = 1 \ \vee \ (z + \frac{1}{2})^2 = 3i^2$
$z = 1 \ \vee \ z + \frac{1}{2} = i\sqrt{3} \ \vee \ z + \frac{1}{2} = -i\sqrt{3}$
$\underline{\underline{z = 1 \ \vee \ z = -\frac{1}{2} + i\sqrt{3} \ \vee \ z = -\frac{1}{2} - i\sqrt{3}.}}$

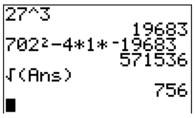
37d $z^3+81z=702$

$z=u+v$ en $81=-3uv$ geeft $u^3+v^3=702$
Uit $-3uv=81$ volgt $v=-\frac{27}{u}$ $\Rightarrow$

$u^3+\left(-\frac{27}{u}\right)^3=702$

$u^3-\frac{19683}{u^3}=702$

$u^6-702u^3-19683=0$

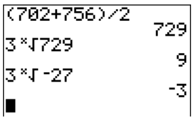


$D=(-702)^2-4\cdot1\cdot-19683=571536\Rightarrow\sqrt{D}=756$

$u^3=\frac{702+756}{2}=729$ is een oplossing

$v^3=702-u^3=702-729=-27$

$z=u+v=\sqrt[3]{729}+\sqrt[3]{-27}=9-3=6$



De staartdeling: $z-6/z^3+81z-702\backslash z^2+6z+117$

$$\begin{array}{l} z^3-6z^2 \quad - \\ 6z^2+81z-702 \\ 6z^2-36z \quad - \\ 117z-702 \\ 117z-702 \quad - \\ 0 \end{array}$$

$z^3+81z-702=0$

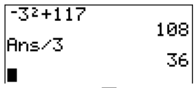
$(z-6)(z^2+6z+117)=0$

$z=6 \vee z^2+6z+117=0$

$z=6 \vee (z+3)^2-9+117=0$

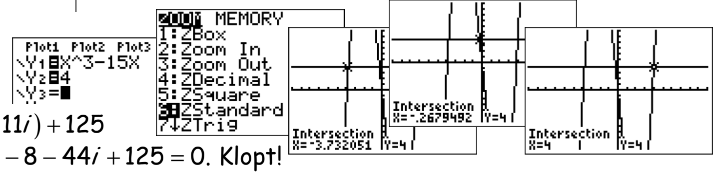
$z=6 \vee (z+3)^2=-108$

$z=6 \vee (z+3)^2=108i^2$



$z=6 \vee z+3=6i\sqrt{3} \vee z+3=-6i\sqrt{3}$

$\underline{z=6 \vee z=-3+6i\sqrt{3} \vee z=-3-6i\sqrt{3}.}$

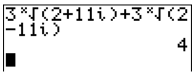


38a $z^3-15z=4$ (intersect en z geheel) $\Rightarrow z=4$.

38b $\sqrt[3]{2+11i}^{\,6}-4\sqrt[3]{2+11i}^{\,3}+125=(2+11i)^2-4\cdot(2+11i)+125$
$=4+22i+22i-121-8-44i+125=0$. Klopt!

$z=u+v$ en $-15=-3uv$ geeft $u^3+v^3=4\Rightarrow v^3=4-u^3=4-(2+11i)=2-11i\Rightarrow v=\sqrt[3]{2-11i}$.

38c $z=u+v=\sqrt[3]{2+11i}+\sqrt[3]{2-11i}=4.$



Dus $z=4$ is een oplossing van $z^3-15z=4$.

39a $z^3+6z^2+12z+9=0$

Stel $z=y-\frac{1}{3}\cdot6=y-2$

$(y-2)^3+6(y-2)^2+12(y-2)+9=0$

$y^3-6y^2+12y-8+6y^2-24y+24+12y-24+9=0$

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$

$(a-b)^2=a^2-2ab+b^2$

$y^3+1=0$

$y^3=-1$

$y=-1$

$z=y-2=-1-2=-3$ is een oplossing

Nu de staartdeling (zie hiernaast)

$z^3+3z+3=0$ $\quad z+3/z^3+6z^2+12z+9\backslash z^2+3z+3$

$$\begin{array}{l} z^3-3z^2 \quad - \\ 3z^2+12z+9 \\ 3z^2+9z \quad - \\ 3z+9 \\ 3z+9 \quad - \\ 0 \end{array}$$

$(z+3)(z^2+3z+3)=0$

$z=-3 \vee z^2+3z+3=0$

$z=-3 \vee (z+\frac{3}{2})^2-\frac{9}{4}+\frac{12}{4}=0$

$z=-3 \vee (z+\frac{3}{2})^2=-\frac{3}{4}$

$z=-3 \vee (z+\frac{3}{2})^2=\frac{3}{4}i^2$

$z=-3 \vee z+\frac{3}{2}=\frac{1}{2}i\sqrt{3} \vee z+\frac{3}{2}=-\frac{1}{2}i\sqrt{3}$

$\underline{z=-3 \vee z=-\frac{3}{2}+\frac{1}{2}i\sqrt{3} \vee z=-\frac{3}{2}-\frac{1}{2}i\sqrt{3}.}$

39b $z^3-12z^2-392=0$

Stel $z=y-\frac{1}{3}\cdot-12=y+4$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$(y+4)^3-12(y+4)^2-392=0$

$y^3+12y^2+48y+64-12y^2-96y-192-392=0$

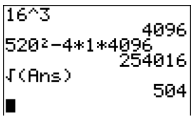
$y^3-48y=520$

$y=u+v$ en $-48=-3uv$ geeft $u^3+v^3=520$
Uit $-3uv=-48$ volgt $v=\frac{16}{u}$ $\Rightarrow$

$u^3+\left(\frac{16}{u}\right)^3=520$

$u^3+\frac{4096}{u^3}=520$

$u^6-520u^3+4096=0$



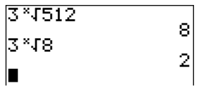
$D=(-520)^2-4\cdot1\cdot4096=254016\Rightarrow\sqrt{D}=504$

$u^3=\frac{520+504}{2}=512$ is een oplossing

$v^3=520-u^3=520-512=8$

$y=u+v=\sqrt[3]{512}+\sqrt[3]{8}=8+2=10$

$z=y+4=10+4=14$



De staartdeling: $z-14/z^3-12z^2-392\backslash z^2+2z+28$

$$\begin{array}{l} z^3-14z^2 \quad - \\ 2z^2 \quad -392 \\ 2z^2-28z \quad - \\ 28z-392 \\ 28z-392 \quad - \\ 0 \end{array}$$

$z^3-12z^2-392=0$

$(z-14)(z^2+2z+28)=0$

$z=14 \vee z^2+2z+28=0$

$z=14 \vee (z+1)^2-1+28=0$

$z=14 \vee (z+1)^2=-27$

$z=14 \vee (z+1)^2=27i^2$

$z=14 \vee z+1=3i\sqrt{3} \vee z+1=-3i\sqrt{3}$

$\underline{z=14 \vee z=-1+3i\sqrt{3} \vee z=-1-3i\sqrt{3}.}$

39c $z^3+3z^2+4z-28=0$

Stel $z=y-\frac{1}{3}\cdot3=y-1$

$(y-1)^3+3(y-1)^2+4(y-1)-28=0$

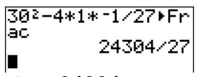
$y^3-3y^2+3y-1+3y^2-6y+3+4y-4-28=0$

$y^3+y=30$

$y=u+v$ en $1=-3uv$ geeft $u^3+v^3=30$
Uit $-3uv=1$ volgt $v=-\frac{1}{3u}$ $\Rightarrow$

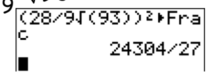
$u^3+\left(-\frac{1}{3u}\right)^3=30\Rightarrow u^3-\frac{1}{27u^3}=30$

$u^6-30u^3-\frac{1}{27}=0$



$D=(-30)^2-4\cdot1\cdot-\frac{1}{27}=\frac{24304}{27}\Rightarrow\sqrt{D}=\frac{28}{9}\sqrt{93}$

$u^3=\frac{30+\frac{28}{9}\sqrt{93}}{2}=15+\frac{14}{9}\sqrt{93}$



$v^3=30-u^3=30-\left(15+\frac{14}{9}\sqrt{93}\right)=15+\frac{14}{9}\sqrt{93}$

$y=u+v=\sqrt[3]{15+\frac{14}{9}\sqrt{93}}+\sqrt[3]{15+\frac{14}{9}\sqrt{93}}=3$

$z=y-1=3-1=2$

de staartdeling en het vervolg op het volgend blad

$z-2/z^3+3z^2+4z-28\backslash z^2+5z+14$
$\underline{z^3-2z^2}\ -$
$5z^2+4z-28$
$\underline{5z^2-10z}\ -$
$14z-28$
$\underline{14z-28}\ -$
0

$z^3+3z^2+4z-28=0$
$(z-2)(z^2+5z+14)=0$
$z=2 \ \vee\ z^2+5z+14=0$
$z=2 \ \vee\ (z+\frac{5}{2})^2-\frac{25}{4}+14=0$
$z=2 \ \vee\ (z+\frac{5}{2})^2=-\frac{31}{4}$
$z=2 \ \vee\ (z+\frac{5}{2})^2=\frac{31}{4}i^2$
$z=2 \ \vee\ z+\frac{5}{2}=\frac{1}{2}i\sqrt{31} \ \vee\ z+\frac{5}{2}=-\frac{1}{2}i\sqrt{31}$
$z=2 \ \vee\ z=-\frac{5}{2}+\frac{1}{2}i\sqrt{31} \ \vee\ z=-\frac{5}{2}-\frac{1}{2}i\sqrt{31}.$

40a $\left(y-\frac{1}{3}a\right)^3+a\left(y-\frac{1}{3}a\right)^2+b\left(y-\frac{1}{3}a\right)+c=0$
$y^3-ay^2+\frac{1}{3}a^2y-\frac{1}{27}a^3+a\left(y^2-\frac{2}{3}ay+\frac{1}{9}a^2\right)+by-\frac{1}{3}ab+c=0$
$y^3-\cancel{ay^2}+\frac{1}{3}a^2y-\frac{1}{27}a^3+\cancel{ay^2}-\frac{2}{3}a^2y+\frac{1}{9}a^3+by-\frac{1}{3}ab+c=0$
$y^3-\frac{1}{3}a^2y+by+\frac{2}{27}a^3-\frac{1}{3}ab+c=0$
$y^3+\left(b-\frac{1}{3}a^2\right)y=-\frac{2}{27}a^3+\frac{1}{3}ab-c.$

40b Neem $p=b-\frac{1}{3}a^2$ en $q=-\frac{2}{27}a^3+\frac{1}{3}ab-c$ dan $z^3+az^2+bz+c=0$ omgeschreven tot $y^3+pz=q$.
Stel $y=u+v$ en $p=-3uv$. Dit geeft $u^3+v^3=q\ \bullet$. Uit $p=-3uv$ volgt $v=-\frac{p}{3u}$ invullen in $\bullet$.
$u^3+\left(-\frac{p}{3u}\right)^3=q \Rightarrow u^3-\frac{p^3}{27u^3}=q \Rightarrow u^6-qu^3-\frac{1}{27}p^3=0$. Stel $x=u^3$ dan krijg je $x^2-qx-\frac{1}{27}p^3=0\ \bullet\bullet$ met
$D=q^2+\frac{4}{27}p^3=4\left(\frac{1}{4}q^2+\frac{1}{27}p^3\right)=4(\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3) \Rightarrow \sqrt{D}=2\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}.$
$u^3=x=\frac{q+2\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}}{2}=\frac{1}{2}q+\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}$ is een oplossing van $\bullet\bullet \Rightarrow v^3=q-u^3=\frac{1}{2}q-\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}.$
$y=u+v=\sqrt[3]{\frac{1}{2}q+\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}}+\sqrt[3]{\frac{1}{2}q-\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}}.$
Dus $z=y-\frac{1}{3}a=-\frac{1}{3}a+\sqrt[3]{\frac{1}{2}q+\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}}+\sqrt[3]{\frac{1}{2}q-\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}}$ met
$p=b-\frac{1}{3}a^2$ en $q=-\frac{2}{27}a^3+\frac{1}{3}ab-c$ is een reële oplossing van $z^3+az^2+bz+c=0.$

41 Stel $T=\sqrt[3]{\frac{1}{2}q+\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}}+\sqrt[3]{\frac{1}{2}q-\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}}.$
Dit geeft $y=T$ is een oplossing van $y^3+py=q$, dus $T^3+pT=q$ ofwel $T^3+pT-q=0\ ■$.
Ontbinden van $y^3+py-q=0$ geeft dan: (zie de staartdeling hieronder)

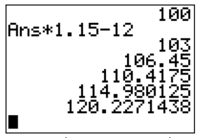
$(y-T)(y^2+Ty+p+T^3)=0$
$y=T \ \vee\ y^2+Ty+p+T^3=0$
$y=T \ \vee\ (y+\frac{1}{2}T)^2-\frac{1}{4}T^2+p+T^3=0$
$y=T \ \vee\ (y+\frac{1}{2}T)^2=\frac{1}{4}T^2-p-T^3$
$y=T \ \vee\ (y+\frac{1}{2}T)^2=-\frac{3}{4}T^2-p$
$y=T \ \vee\ (y+\frac{1}{2}T)^2=(\frac{3}{4}T^2+p)i^2$
$y=T \ \vee\ (y+\frac{1}{2}T)^2=\frac{1}{4}i^2(3T^2+4p)$
$y=T \ \vee\ y+\frac{1}{2}T=\frac{1}{2}i\sqrt{3T^2+4p} \ \vee\ y+\frac{1}{2}T=-\frac{1}{2}i\sqrt{3T^2+4p}$
$y=T \ \vee\ y=-\frac{1}{2}T+\frac{1}{2}i\sqrt{3T^2+4p} \ \vee\ y=-\frac{1}{2}T+-\frac{1}{2}i\sqrt{3T^2+4p}$

$y-T/y^3 \quad +py \quad -q\backslash y^2+Ty+p+T^2$
$\underline{y^3-Ty^2}\ -$
$Ty^2+\ py \quad -q$
$\underline{Ty^2-T^2y}\ -$
$(p+T^2)y \quad -q$
$\underline{(p+T^2)y-pT-T^3}\ -$
$T^3+pT-q=0\ ■$

Dus $z=y-\frac{1}{3}a=-\frac{1}{3}a+T \ \vee\ y=-\frac{1}{3}a-\frac{1}{2}T+\frac{1}{2}i\sqrt{3T^2+4p} \ \vee\ y=-\frac{1}{3}a-\frac{1}{2}T+-\frac{1}{2}i\sqrt{3T^2+4p}$
Dus de oplossing van $z^3+az^2+bz=0$ in $\mathbb{C}$ is:
$z=y-\frac{1}{3}a=-\frac{1}{3}a+T \ \vee\ y=-\frac{1}{3}a+\frac{-T+i\sqrt{3T^2+4p}}{2} \ \vee\ y=-\frac{1}{3}a+\frac{-T-i\sqrt{3T^2+4p}}{2}$
met $T=\sqrt[3]{\frac{1}{2}q+\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}}+\sqrt[3]{\frac{1}{2}q-\sqrt{\left(\frac{1}{2}q\right)^2+\left(\frac{1}{3}p\right)^3}},\quad p=b-\frac{1}{3}a^2$ en $q=-\frac{2}{27}a^3+\frac{1}{3}ab-c.$

42a $u_n=1{,}15u_{n-1}-12$ is te schrijven als $u_n-u_{n-1}=0{,}15u_{n-1}-12.$
In de laatste vorm komt het verschil (= differentie) u_n-u_{n-1} voor. Vandaar de naam differentievergelijking.

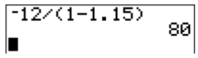
42b De differentievergelijking geeft het verband tussen u_n en zijn direct voorafgaande term u_{n-1}.
Daarom is de differentievergelijking van de eerste orde.



42c $u_5 \approx 120$ en $u_{10} \approx 161$.

42d Bij de recursieve formule $u_n = a \cdot u_{n-1} + b$ met beginterm u_0 hoort de directe formule $u_n = \frac{b}{1-a} + a^n \left(u_0 - \frac{b}{1-a}\right)$.

$u_0 = 100,\ b = -12$ en $a = 1{,}15 \Rightarrow u_n = \frac{-12}{1-1{,}15} + 1{,}15^n \left(u_0 - \frac{-12}{1-1{,}15}\right) = 80 + 1{,}15^n(100-80) = 80 + 20 \cdot 1{,}15^n$.



43a $u_n = g^n \Rightarrow u_{n-1} = g^{n-1}$ en $u_{n-2} = g^{n-2}$.
Invullen in $u_n = 2u_{n-1} + 3u_{n-2} \Rightarrow g^n = 2g^{n-1} + 3g^{n-2}$ (delen door g^{n-2}) $\Rightarrow g^2 = 2g + 3$.

43b $g^2 = 2g + 3 \Rightarrow g^2 - 2g - 3 = 0 \Rightarrow (g+1)(g-3) = 0 \Rightarrow g = -1 \ \vee \ g = 3$.

43c $u_n = A \cdot (-1)^n + B \cdot 3^n$ met $u_0 = 1 \Rightarrow u_0 = A \cdot (-1)^0 + B \cdot 3^0 = A \cdot 1 + B \cdot 1 = A + B = 1$ **(1)**.

43d $u_n = A \cdot (-1)^n + B \cdot 3^n$ met $u_1 = 5 \Rightarrow u_1 = A \cdot (-1)^1 + B \cdot 3^1 = A \cdot -1 + B \cdot 3 = -A + 3B = 5$ **(2)**.

43e $\begin{cases} A + B = 1 & \textbf{(1)} \\ -A + 3B = 5 & \textbf{(2)}\ + \end{cases}$

$4B = 6 \Rightarrow B = \frac{6}{4} = 1\frac{1}{2}$ in **(1)** $\Rightarrow A + 1\frac{1}{2} = 1 \Rightarrow A = -\frac{1}{2}$.

Dus de formule: $u_n = 2u_{n-1} + 3u_{n-2}$ met $u_0 = 1$ en $u_1 = 5$ heeft als directe formule $u_n = -\frac{1}{2} \cdot (-1)^n + 1\frac{1}{2} \cdot 3^n$.

43f $u_n = -\frac{1}{2} \cdot (-1)^n + 1\frac{1}{2} \cdot 3^n$ substitueren in $u_n = 2u_{n-1} + 3u_{n-2}$ (met $u_0 = 1$ en $u_1 = 5$)

$-\frac{1}{2} \cdot (-1)^n + 1\frac{1}{2} \cdot 3^n = 2\left(-\frac{1}{2} \cdot (-1)^{n-1} + 1\frac{1}{2} \cdot 3^{n-1}\right) + 3\left(-\frac{1}{2} \cdot (-1)^{n-2} + 1\frac{1}{2} \cdot 3^{n-2}\right)$

$-\frac{1}{2} \cdot (-1)^n + 1\frac{1}{2} \cdot 3^n = -2 \cdot (-1)^{n-1} + 3 \cdot 3^{n-1} - 1\frac{1}{2} \cdot (-1)^{n-2} + 4\frac{1}{2} \cdot 3^{n-2}$

$-\frac{1}{2} \cdot (-1)^n + 1\frac{1}{2} \cdot 3^n = -2 \cdot (-1)^n \cdot (-1)^{-1} + 3 \cdot 3^n \cdot 3^{-1} - 1\frac{1}{2} \cdot (-1)^n \cdot (-1)^{-2} + 4\frac{1}{2} \cdot 3^n \cdot 3^{-2}$

$-\frac{1}{2} \cdot (-1)^n + 1\frac{1}{2} \cdot 3^n = -2 \cdot (-1)^n \cdot \frac{1}{-1} + 3 \cdot 3^n \cdot \frac{1}{3} - 1\frac{1}{2} \cdot (-1)^n \cdot \frac{1}{(-1)^2} + 4\frac{1}{2} \cdot 3^n \cdot \frac{1}{3^2}$

$-\frac{1}{2} \cdot (-1)^n + 1\frac{1}{2} \cdot 3^n = 2 \cdot (-1)^n + 3^n - 1\frac{1}{2} \cdot (-1)^n + \frac{1}{2} \cdot 3^n$

$-\frac{1}{2} \cdot (-1)^n + 1\frac{1}{2} \cdot 3^n = -\frac{1}{2} \cdot (-1)^n + 1\frac{1}{2} \cdot 3^n$. Klopt! ($u_0 = 1$ en $u_1 = 5$ klopt ook, zie 43cde).

44a $u_n = 2u_{n-1} + 8u_{n-2}$ met $u_0 = 1$ en $u_1 = 2$.
Substitueer $u_n = g^n$ in $u_n = 2u_{n-1} + 8u_{n-2}$.
$g^n = 2g^{n-1} + 8g^{n-2}$
$g^2 = 2g + 8$
$g^2 - 2g - 8 = 0$
$(g+2)(g-4) = 0$
$g = -2 \ \vee \ g = 4$.
Stel nu $u_n = A \cdot (-2)^n + B \cdot 4^n$.

$u_0 = 1 \Rightarrow A \cdot (-2)^0 + B \cdot 4^0 = A \cdot 1 + B \cdot 1 = A + B = 1$ **(1)**
$u_1 = 2 \Rightarrow A \cdot (-2)^1 + B \cdot 4^1 = A \cdot -2 + B \cdot 4 = -2A + 4B = 2$ **(2)**

$\begin{cases} A + B = 1 \ \textbf{(1)} & |2| \\ -2A + 4B = 2 \ \textbf{(2)} & |1| \end{cases} \Rightarrow \begin{cases} 2A + 2B = 2 & \textbf{(3)} \\ -2A + 4B = 2 & \textbf{(2)}\ + \end{cases}$

$6B = 4 \Rightarrow B = \frac{4}{6} = \frac{2}{3}$ **(4)**

(4) in **(1)** $\Rightarrow A + \frac{2}{3} = 1 \Rightarrow A = \frac{1}{3}$.
Dus $u_n = \frac{1}{3} \cdot (-2)^n + \frac{2}{3} \cdot 4^n$.

44b $x_n = 3x_{n-1} + 4x_{n-2}$ met $x_0 = 2$ en $x_1 = 6$.
Substitueer $x_n = g^n$ in $x_n = 3x_{n-1} + 8x_{n-2}$.
$g^n = 3g^{n-1} + 4g^{n-2}$
$g^2 = 3g + 4$
$g^2 - 3g - 4 = 0$
$(g+1)(g-4) = 0$
$g = -1 \ \vee \ g = 4$.
Dus $x_n = A \cdot (-1)^n + B \cdot 4^n$.

$x_0 = 2 \Rightarrow A \cdot (-1)^0 + B \cdot 4^0 = A \cdot 1 + B \cdot 1 = A + B = 2$ **(1)**
$x_1 = 6 \Rightarrow A \cdot (-1)^1 + B \cdot 4^1 = A \cdot -1 + B \cdot 4 = -A + 4B = 6$ **(2)**

$\begin{cases} A + B = 2 & \textbf{(1)} \\ -A + 4B = 6 & \textbf{(2)}\ + \end{cases}$

$5B = 8 \Rightarrow B = \frac{8}{5}$ **(3)**

(3) in **(1)** $\Rightarrow A + \frac{8}{5} = 2 \Rightarrow A = \frac{2}{5}$.
Dus $x_n = \frac{2}{5} \cdot (-1)^n + \frac{8}{5} \cdot 4^n$.

44c $v_n = 5v_{n-1} - 6v_{n-2}$ met $v_0 = 1$ en $v_1 = 3$.
Substitueer $v_n = g^n$ in $v_n = 5v_{n-1} - 6v_{n-2}$.
$g^n = 5g^{n-1} - 5g^{n-2}$
$g^2 = 5g - 5$
$g^2 - 5g + 6 = 0$
$(g-2)(g-3) = 0$
$g = 2 \ \vee \ g = 3$.
Dus $v_n = A \cdot 2^n + B \cdot 3^n$.

$v_0 = 1 \Rightarrow A \cdot 2^0 + B \cdot 3^0 = A \cdot 1 + B \cdot 1 = A + B = 1$ **(1)**
$v_1 = 2 \Rightarrow A \cdot 2^1 + B \cdot 3^1 = A \cdot 2 + B \cdot 3 = 2A + 3B = 3$ **(2)**

$\begin{cases} A + B = 1 \ \textbf{(1)} & |2| \\ 2A + 3B = 3 \ \textbf{(2)} & |1| \end{cases} \Rightarrow \begin{cases} 2A + 2B = 2 & \textbf{(3)} \\ 2A + 3B = 3 & \textbf{(2)}\ - \end{cases}$

$-B = -1 \Rightarrow B = 1$ **(4)**

(4) in **(1)** $\Rightarrow A + 1 = 1 \Rightarrow A = 0$.
Dus $v_n = 3^n$.

45a $u_n = u_{n-1} + u_{n-2}$ met $u_0 = 0$ en $u_1 = 1$ geeft de rij 0, 1, 1, 2, 3, 5, 8, ... en dit is, afgezien van de eerste term, de rij van Fibonacci.

45b Substitueer $u_n = g^n$ in $u_n = u_{n-1} + u_{n-2}$.
$g^n = g^{n-1} + g^{n-2}$
$g^2 = g + 1$
$g^2 - g - 1 = 0$
$D = 1^2 - 4 \cdot 1 \cdot -1 = 5 \Rightarrow \sqrt{D} = 5$
$g = \frac{1+\sqrt{5}}{2} \quad \vee \quad g = \frac{1-\sqrt{5}}{2}$.
Dus $u_n = A \cdot (\frac{1+\sqrt{5}}{2})^n + B \cdot (\frac{1-\sqrt{5}}{2})^n$.

45c $u_0 = 0 \Rightarrow A \cdot (\frac{1+\sqrt{5}}{2})^0 + B \cdot (\frac{1-\sqrt{5}}{2})^0 = A + B = 0$
$u_1 = 1 \Rightarrow A \cdot (\frac{1+\sqrt{5}}{2})^1 + B \cdot (\frac{1-\sqrt{5}}{2})^1 = (\frac{1}{2} + \frac{1}{2}\sqrt{5})A + (\frac{1}{2} - \frac{1}{2}\sqrt{5})B = 1$
$\begin{cases} A + B = 0 \ \textbf{(1)} \\ (\frac{1}{2} + \frac{1}{2}\sqrt{5})A + (\frac{1}{2} - \frac{1}{2}\sqrt{5})B = 1 \ \textbf{(2)} \end{cases}$
Uit **(1)** volgt $B = -A$ **(3)**
(3) in **(2)** geeft $(\frac{1}{2} + \frac{1}{2}\sqrt{5})A - (\frac{1}{2} - \frac{1}{2}\sqrt{5})A = 1$
$\frac{1}{2}A + \frac{1}{2}A\sqrt{5} - \frac{1}{2}A + \frac{1}{2}A\sqrt{5} = 1$
$A\sqrt{5} = 1 \Rightarrow A = \frac{1}{\sqrt{5}}$ in **(2)** $\Rightarrow \frac{1}{\sqrt{5}} + B = 0 \Rightarrow B = -\frac{1}{\sqrt{5}}$.
Dus $A = \frac{1}{\sqrt{5}}$ en $B = -\frac{1}{\sqrt{5}}$.

46a $u_n = A \cdot (g_1)^n + B \cdot (g_2)^n$ substitueren in $u_n = a \cdot u_{n-1} + b \cdot u_{n-2}$ geeft:
$A \cdot (g_1)^n + B \cdot (g_2)^n = a \cdot \left(A \cdot (g_1)^{n-1} + B \cdot (g_2)^{n-1}\right) + b \cdot \left(A \cdot (g_1)^{n-2} + B \cdot (g_2)^{n-2}\right)$
$A(g_1)^n + B(g_2)^n = aA(g_1)^{n-1} + aB(g_2)^{n-1} + bA(g_1)^{n-2} + bB(g_2)^{n-2}$
$A(g_1)^n + B(g_2)^n - aA(g_1)^{n-1} - aB(g_2)^{n-1} - bA(g_1)^{n-2} - bB(g_2)^{n-2} = 0$
$A(g_1)^{n-2}\left(g_1^2 - ag_1 - b\right) + B(g_2)^{n-2}\left(g_2^2 - ag_2 - b\right) = 0$.

46b $g_1^2 - ag_1 - b = 0$ en $g_2^2 - ag_2 - b = 0$ (want gegeven is dat g_1 en g_2 oplossingen zijn van $g^2 - ag - b = 0$) geeft dan:
$A(g_1)^{n-2} \cdot 0 + B(g_2)^{n-2} \cdot 0 = 0$
$0 + 0 = 0$.
Klopt!

47a $u_n = 4u_{n-1} - 4u_{n-2}$ met $u_0 = 1$ en $u_1 = 3$.
Substitueer $u_n = g^n$ in $u_n = 4u_{n-1} - 4u_{n-2}$.
$g^n = 4g^{n-1} - 4g^{n-2}$
$g^2 = 4g - 4$
$g^2 - 4g + 4 = 0$
$(g-2)(g-2) = 0$
$g = 2 \quad \vee \quad g = 2$.
Dus $u_n = A \cdot 2^n + B \cdot 2^n$.
$u_0 = 1 \Rightarrow A \cdot 2^0 + B \cdot 2^0 = A \cdot 1 + B \cdot 1 = A + B = 1$ **(1)**
$u_1 = 3 \Rightarrow A \cdot 2^1 + B \cdot 2^1 = A \cdot 2 + B \cdot 2 = 2A + 2B = 3$ **(2)**
$\begin{cases} A + B = 1 \ \textbf{(1)} \\ 2A + 2B = 3 \ \textbf{(2)} \end{cases} \begin{vmatrix} 2 \\ 1 \end{vmatrix} \Rightarrow \begin{cases} 2A + 2B = 2 \ \textbf{(3)} \\ \underline{2A + 2B = 3} \ - \ \textbf{(2)} \end{cases}$
$0 = -1$ (kan niet)
Dus op deze manier geen directe formule af te leiden.

47b $v_n = 5v_{n-1} - 6v_{n-2}$ met $v_0 = 1$ en $v_1 = 3$.
Substitueer $v_n = g^n$ in $v_n = 5v_{n-1} - 6v_{n-2}$.
$g^n = 2g^{n-1} - 4g^{n-2}$
$g^2 = 2g - 4$
$g^2 - 2g + 4 = 0$
$D = 2^2 - 4 \cdot 1 \cdot 4 = 4 - 16 - 12 < 0$ (je krijgt dus complexe getallen).

48a $u_n = 2\sqrt{3} \cdot u_{n-1} - 4u_{n-2}$ met $u_0 = 4$ en $u_1 = \sqrt{3}$.
Substitueer $u_n = g^n$ in $u_n = 2\sqrt{3} \cdot u_{n-1} - 4u_{n-2}$.
$g^2 - 2\sqrt{3} \cdot g + 4 = 0$
$D = \left(-2\sqrt{3}\right)^2 - 4 \cdot 1 \cdot 4 = -4 = 4i^2 \Rightarrow \sqrt{D} = 2i$
$g = \frac{2\sqrt{3} \pm 2i}{2}$
$g = \sqrt{3} + i \quad \vee \quad g = \sqrt{3} - i$.
$\left|\sqrt{3} + i\right| = \sqrt{4} = 2$ en $\arg(\sqrt{3} + i) = \frac{1}{6}\pi$.
Dus $u_n = \left(A\cos(\frac{1}{6}\pi n) + B\sin(\frac{1}{6}\pi n)\right) \cdot 2^n$.
$u_0 = 4 \Rightarrow (A \cdot 1 + B \cdot 0) \cdot 1 = A = 4$ **(1)**
$u_1 = \sqrt{3} \Rightarrow \left(A \cdot \frac{1}{2}\sqrt{3} + B \cdot \frac{1}{2}\right) \cdot 2 = A\sqrt{3} + B = \sqrt{3}$ **(2)**
(1) in **(2)** $\Rightarrow 4\sqrt{3} + B = \sqrt{3} \Rightarrow B = -3\sqrt{3}$.
Dus $u_n = \left(4\cos(\frac{1}{6}\pi n) - 3\sqrt{3}\sin(\frac{1}{6}\pi n)\right) \cdot 2^n$.

48b $v_n = -4v_{n-1} - 16v_{n-2}$ met $v_0 = 1$ en $v_1 = 2$.
Substitueer $v_n = g^n$ in $v_n = -4v_{n-1} - 16v_{n-2}$.
$g^2 + 4g + 16 = 0$
$D = 4^2 - 4 \cdot 1 \cdot 16 = -48 = 48i^2 \Rightarrow \sqrt{D} = 4i\sqrt{3}$
$g = \frac{-4 \pm 4i\sqrt{3}}{2}$
$g = -2 + 2i\sqrt{3} \quad \vee \quad g = -2 - 2i\sqrt{3}$.
$\left|-2 + 2i\sqrt{3}\right| = \sqrt{16} = 4$ en $\arg(-2 + 2i\sqrt{3}) = \frac{2}{3}\pi$.
Dus $v_n = \left(A\cos(\frac{2}{3}\pi n) + B\sin(\frac{2}{3}\pi n)\right) \cdot 4^n$.
$v_0 = 1 \Rightarrow (A \cdot 1 + B \cdot 0) \cdot 1 = A = 1$ **(1)**
$v_1 = 2 \Rightarrow \left(A \cdot -\frac{1}{2} + B \cdot \frac{1}{2}\sqrt{3}\right) \cdot 4 = -2A + 2B\sqrt{3} = 2$ **(2)**
(1) in **(2)** $\Rightarrow -2 + 2B\sqrt{3} = 2 \Rightarrow 2B\sqrt{3} = 4 \Rightarrow$
$B = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$.
Dus $v_n = \left(\cos(\frac{2}{3}\pi n) + \frac{2}{3}\sqrt{3}\sin(\frac{2}{3}\pi n)\right) \cdot 4^n$.

49a Substitueer $x_n = g^n$ in $x_n = 2x_{n-1} - x_{n-2}$.
$g^2 - 2g + 1 = 0$
$(g-1)(g-1) = 0$
$g = 1 \ \vee \ g = 1.$
Dus $x_n = (A + Bn) \cdot 1^n = A + Bn.$
$x_0 = 3 \Rightarrow A + B \cdot 0 = A = 3$ **(1)**
$x_1 = 5 \Rightarrow A + B \cdot 1 = A + B = 5$ **(2)**
(1) in **(2)** $\Rightarrow 3 + B = 5 \Rightarrow B = 2.$
Dus $x_n = 3 + 2n.$

49b Substitueer $y_n = g^n$ in $y_n = -6y_{n-1} - 9y_{n-2}$.
$g^2 + 6g + 9 = 0$
$(g+3)(g+3) = 0$
$g = -3 \ \vee \ g = -3.$
Dus $y_n = (A + Bn) \cdot (-3)^n$.
$y_0 = 1 \Rightarrow (A + B \cdot 0) \cdot 1 = A = 1$ **(1)**
$y_1 = 2 \Rightarrow (A + B \cdot 1) \cdot -3 = -3A - 3B = 2$ **(2)**
(1) in **(2)** $\Rightarrow -3 - 3B = 2 \Rightarrow -3B = 5 \Rightarrow B = -\frac{5}{3}.$
Dus $y_n = \left(1 - 1\frac{2}{3}n\right) \cdot (-3)^n$.

50a Substitueer $u_n = g^n$ in $u_n = 7u_{n-1} - 10u_{n-2}$.
$g^2 - 7g + 10 = 0$
$(g-2)(g-5) = 0$
$g = 2 \ \vee \ g = 5.$
Dus $u_n = A \cdot 2^n + B \cdot 5^n.$
$u_0 = 1 \Rightarrow A \cdot 1 + B \cdot 1 = A + B = 1$ **(1)**
$u_1 = 3 \Rightarrow A \cdot 2 + B \cdot 5 = 2A + 5B = 3$ **(2)**
$\begin{cases} A + B = 1 \ \textbf{(1)} \\ 2A + 5B = 3 \ \textbf{(2)} \end{cases} \begin{vmatrix} 2 \\ 1 \end{vmatrix} \Rightarrow \begin{cases} 2A + 2B = 2 \quad \textbf{(3)} \\ \underline{2A + 5B = 3} \ _{+} \ \textbf{(2)} \end{cases}$
$-3B = -1 \Rightarrow B = \frac{1}{3}$ **(4)**
(4) in **(1)** $\Rightarrow A + \frac{1}{3} = 1 \Rightarrow A = \frac{2}{3}.$
Dus $u_n = \frac{2}{3} \cdot 2^n + \frac{1}{3} \cdot 5^n.$

50b Substitueer $v_n = g^n$ in $v_n = 3v_{n-1} - 2\frac{1}{4}v_{n-2}$.
$g^2 - 3g + 2\frac{1}{4} = 0$
$(g - 1\frac{1}{2})^2 = 0$
$g = 1\frac{1}{2} \ \vee \ g = 1\frac{1}{2}.$
Dus $v_n = (A + Bn) \cdot 1\frac{1}{2}^n$.
$v_0 = 2 \Rightarrow (A + B \cdot 0) \cdot 1 = A = 2$ **(1)**
$v_1 = 4 \Rightarrow (A + B \cdot 1) \cdot 1\frac{1}{2} = 1\frac{1}{2}A + 1\frac{1}{2}B = 4$ **(2)**
(2) in **(1)** $\Rightarrow 3 + 1\frac{1}{2}B = 4 \Rightarrow 1\frac{1}{2}B = 1 \Rightarrow 3B = 2 \Rightarrow B = \frac{2}{3}.$
Dus $v_n = \left(2 + \frac{2}{3}n\right) \cdot 1\frac{1}{2}^n$.

50c Substitueer $w_n = g^n$ in $w_n = -6w_{n-1} - 36w_{n-2}$.
$g^2 + 6g + 36 = 0$
$D = (-6)^2 - 4 \cdot 1 \cdot 36 = -3 \cdot 36 = 3 \cdot 36i^2 \Rightarrow \sqrt{D} = 6i\sqrt{3}$
$g = \frac{-6 \pm 6i\sqrt{3}}{2}$
$g = -3 + 3i\sqrt{3} \ \vee \ g = -3 - 3i\sqrt{3}.$
$\left|-3 + 3i\sqrt{3}\right| = \sqrt{36} = 6$ en $\arg(-3 + 3i\sqrt{3}) = \frac{2}{3}\pi.$
Dus $w_n = \left(A\cos(\frac{2}{3}\pi n) + B\sin(\frac{2}{3}\pi n)\right) \cdot 6^n.$
$w_0 = 0 \Rightarrow (A \cdot 1 + B \cdot 0) \cdot 1 = A = 0$ **(1)**
$w_1 = 1 \Rightarrow \left(A \cdot -\frac{1}{2} + B \cdot \frac{1}{2}\sqrt{3}\right) \cdot 6 = -3A + 3B\sqrt{3} = 1$ **(2)**
(1) in **(2)** $\Rightarrow 3B\sqrt{3} = 1 \Rightarrow B = \frac{1}{3\sqrt{3}} = \frac{1}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{1}{9}\sqrt{3}.$
Dus $w_n = \frac{1}{9}\sqrt{3}\sin(\frac{2}{3}\pi n) \cdot 6^n.$

50d Substitueer $x_n = g^n$ in $x_n = -\frac{1}{4}x_{n-2}$.
$g^2 + \frac{1}{4} = 0$
$g^2 = \frac{1}{4}i^2$
$g = \frac{1}{2}i \ \vee \ g = -\frac{1}{2}i.$
$\left|\frac{1}{2}i\right| = \frac{1}{2}$ en $\arg(\frac{1}{2}i) = \frac{1}{2}\pi.$
Dus $x_n = \left(A\cos(\frac{1}{2}\pi n) + B\sin(\frac{1}{2}\pi n)\right) \cdot \left(\frac{1}{2}\right)^n$.
$x_0 = 16 \Rightarrow (A \cdot 1 + B \cdot 0) \cdot 1 = A = 16$ **(1)**
$x_1 = 12 \Rightarrow (A \cdot 0 + B \cdot 1) \cdot \frac{1}{2} = \frac{1}{2}B = 12 \Rightarrow B = 24$ **(2)**
Dus $x_n = \left(16\cos(\frac{1}{2}\pi n) + 24\sin(\frac{1}{2}\pi n)\right) \cdot \left(\frac{1}{2}\right)^n$.

51a $u_n = (A + Bn) \cdot g^n$ substitueren in $u_n = a \cdot u_{n-1} + b \cdot u_{n-2}$ geeft:
$(A + Bn) \cdot g^n = a \cdot (A + B(n-1)) \cdot g^{n-1} + b \cdot (A + B(n-2)) \cdot g^{n-2}$
$Ag^n + Bng^n = aAg^{n-1} + aBng^{n-1} - aBg^{n-1} + bAg^{n-2} + bBng^{n-2} - 2bBg^{n-2}$
$Ag^n - aAg^{n-1} - bAg^{n-2} + Bng^n - aBng^{n-1} - bBng^{n-2} + aBg^{n-1} + 2bBg^{n-2} = 0$
$Ag^2 - aAg - bA + Bng^2 - aBng - bBn + aBg + 2bB = 0$
$A\left(g^2 - ag - b\right) + B\left(ng^2 - ang - bn + ag + 2b\right) = 0$
$A\left(g^2 - ag - b\right) + B\left(n(g^2 - ag - b) + ag + 2b\right) = 0$ ***

51b Van de karakteristieke vergelijking $g^2 - ag - b = 0$ is $D = 0 \Rightarrow g = \frac{a}{2 \cdot 1} = \frac{1}{2}a.$
Substitutie van $g = \frac{1}{2}a$ in $g^2 - ag - b = 0$ geeft:
$\left(\frac{1}{2}a\right)^2 - a \cdot \frac{1}{2}a - b = 0 \Rightarrow \frac{1}{4}a^2 - \frac{1}{2}a^2 - b = 0 \Rightarrow -\frac{1}{4}a^2 - b = 0 \Rightarrow -\frac{1}{4}a^2 = b.$
Substitutie van $g^2 - ag - b = 0, g = \frac{1}{2}a$ en $b = -\frac{1}{4}a^2$ in *** geeft:
$A \cdot 0 + B\left(n \cdot 0 + a \cdot \frac{1}{2}a + 2 \cdot -\frac{1}{4}a^2\right) = 0 \Rightarrow 0 + B\left(0 + \frac{1}{2}a^2 - \frac{1}{2}a^2\right) = 0 \Rightarrow 0 + B \cdot 0 = 0 \Rightarrow 0 = 0.$ Klopt!

52a In $x_n = 2x_{n-1} + 3y_{n-1}$ **(1)** (geldt voor elke n) mag n vervangen worden door $n+1$. Dit geeft $x_{n+1} = 2x_n + 3y_n$ **(3)**.

52b **(2)** in **(3)** $\Rightarrow x_{n+1} = 2x_n + 3(4x_{n-1} + 5y_{n-1}) \Rightarrow x_{n+1} = 2x_n + 12x_{n-1} + 15y_{n-1}$ **(4)**.

52c $x_n = 2x_{n-1} + 3y_{n-1}$ **(1)** $\Rightarrow 3y_{n-1} = x_n - 2x_{n-1}$ in **(1)** $\Rightarrow x_{n+1} = 2x_n + 12x_{n-1} + 5(x_n - 2x_{n-1}) \Rightarrow x_{n+1} = 7x_n + 2x_{n-1}$ **(5)**.

52d In $x_{n+1} = 7x_n + 2x_{n-1}$ **(5)** (geldt voor elke n) mag n vervangen worden door $n-1$. Dit geeft $x_n = 7x_{n-1} + 2x_{n-2}$.
De differentiaalvergelijking is van de twee orde. (verband tussen een term en zijn twee voorafgaande termen)
Substitutie van $x_n = g^n$ in $x_n = 7x_{n-1} + 2x_{n-2}$ geeft $g^2 - 7g - 2 = 0$.
dit is de karakteristieke vergelijking van de differentiaalvergelijking.

53 $2y_{n-1} = x_n - x_{n-1}$
$2y_{n-1} = -20\cdot 2^n + 30\cdot 3^n - (-20\cdot 2^{n-1} + 30\cdot 3^{n-1})$
$2y_{n-1} = -20\cdot 2^n + 30\cdot 3^n + 20\cdot 2^{n-1} - 30\cdot 3^{n-1}$
$2y_n = -20\cdot 2^{n+1} + 30\cdot 3^{n+1} + 20\cdot 2^n - 30\cdot 3^n$
$2y_n = -20\cdot 2^n\cdot 2^1 + 30\cdot 3^n\cdot 3^1 + 20\cdot 2^n - 30\cdot 3^n$
$y_n = -20\cdot 2^n + 15\cdot 3^n\cdot 3 + 10\cdot 2^n - 15\cdot 3^n$
$y_n = -10\cdot 2^n + 30\cdot 3^n$.

54 $x_n = 3x_{n-1} - 2y_{n-1}$ **(1)** $\Rightarrow x_{n+1} = 3x_n - 2y_n$ **(3)** en $2y_{n-1} = -x_n + 3x_{n-1}$ **(4)**.
$y_n = 2x_{n-1} - 2y_{n-1}$ **(2)** substitueren in **(3)** $\Rightarrow x_{n+1} = 3x_n - 2(2x_{n-1} - 2y_{n-1}) \Rightarrow x_{n+1} = 3x_n - 4x_{n-1} + 4y_{n-1}$ **(5)**.
(4) in **(5)** $\Rightarrow x_{n+1} = 3x_n - 4x_{n-1} + 2(-x_n + 3x_{n-1}) \Rightarrow x_{n+1} = x_n + 2x_{n-1} \Rightarrow x_n = x_{n-1} + 2x_{n-2}$ **(6)**.
De karakteristieke vergelijking van **(6)** is $g^2 - g - 2 = 0 \Rightarrow (g-2)(g+1) = 0 \Rightarrow g = 2 \;\vee\; g = -1$.
Dus $x_n = A\cdot 2^n + B\cdot(-1)^n$.
$x_0 = 5 \Rightarrow A\cdot 1 + B\cdot 1 = A + B = 5$ **(7)**
(1) $\Rightarrow x_1 = 3\cdot 5 - 2\cdot 4 = 7 \Rightarrow 2A - B = 7$ **(8)**
$\begin{cases} A + B = 5 & \textbf{(7)} \\ 2A - B = 7 & \textbf{(8)} \end{cases}$ +
$3A = 12 \Rightarrow A = 4$ in **(7)**
$4 + B = 5 \Rightarrow B = 1$.
Dus $x_n = 4\cdot 2^n + (-1)^n$ **(9)**. (hiernaast verder)

(9) in **(3)** $\Rightarrow 4\cdot 2^{n+1} + (-1)^{n+1} = 3\left(4\cdot 2^n + (-1)^n\right) - 2y_n$
$2y_n = -4\cdot 2^{n+1} - (-1)^{n+1} + 12\cdot 2^n + 3\cdot(-1)^n$
$2y_n = -4\cdot 2\cdot 2^n - (-1)\cdot(-1)^n + 12\cdot 2^n + 3\cdot(-1)^n$
$y_n = -4\cdot 2^n + \frac{1}{2}(-1)^n + 6\cdot 2^n + \frac{3}{2}\cdot(-1)^n$
$y_n = 2\cdot 2^n + 2\cdot(-1)^n$.
Dus $x_n = 4\cdot 2^n + (-1)^n$ en $y_n = 2\cdot 2^n + 2\cdot(-1)^n$.

55 $P_n = 2P_{n-1} - 4Q_{n-1}$ **(1)** $\Rightarrow P_{n+1} = 2P_n - 4Q_n$ **(3)** en $4Q_{n-1} = -P_n + 2P_{n-1}$ **(4)**.
$Q_n = P_{n-1} + 6Q_{n-1}$ **(2)** substitueren in **(3)** $\Rightarrow P_{n+1} = 2P_n - 4(P_{n-1} + 6Q_{n-1}) \Rightarrow P_{n+1} = 2P_n - 4P_{n-1} - 24Q_{n-1}$ **(5)**.
(4) in **(5)** $\Rightarrow P_{n+1} = 2P_n - 4P_{n-1} - 6(-P_n + 2P_{n-1}) \Rightarrow P_{n+1} = 8P_n - 16P_{n-1} \Rightarrow P_n = 8P_{n-1} - 16P_{n-2}$ **(6)**.
De karakteristieke vergelijking van **(6)** is $g^2 - 8g + 16 = 0 \Rightarrow (g-4)(g-4) = 0 \Rightarrow g = 4 \;\vee\; g = 4$.
Dus $P_n = (A + Bn)\cdot 4^n$.
$P_0 = 100 \Rightarrow (A + B\cdot 0)\cdot 1 = A = 100$ **(7)**
(1) $\Rightarrow P_1 = 200 - 40 = 160 \Rightarrow$
$P_1 = 160 \Rightarrow (A + B\cdot 1)\cdot 4 = 160 \Rightarrow A + B = 40$ **(8)**
(8) in **(7)** $\Rightarrow 100 + B = 40 \Rightarrow B = -60$.
Dus $P_n = (100 - 60n)\cdot 4^n$ **(9)**. (hiernaast verder)

(9) in **(3)** $\Rightarrow (100 - 60(n+1))\cdot 4^{n+1} = 2((100 - 60n)\cdot 4^n) - 4Q_n$
$4Q_n = -40\cdot 4^{n+1} + 60n\cdot 4^{n+1} + 200\cdot 4^n - 120n\cdot 4^n$
$4Q_n = -40\cdot 4^n\cdot 4 + 60n\cdot 4^n\cdot 4 + 200\cdot 4^n - 120n\cdot 4^n$
$Q_n = -40\cdot 4^n + 60n\cdot 4^n + 50\cdot 4^n - 30n\cdot 4^n$
Dus $Q_n = 10\cdot 4^n + 30n\cdot 4^n$.

56 $K_t = K_{t-1} + 2L_{t-1}$ **(1)** $\Rightarrow K_{t+1} = K_t + 2L_t$ **(3)** en $2L_{t-1} = K_t - K_{t-1}$ **(4)**.
$L_t = K_{t-1} + 3L_{t-1}$ **(2)** substitueren in **(3)** $\Rightarrow K_{t+1} = K_t + 2(K_{t-1} + 3L_{t-1}) \Rightarrow K_{t+1} = K_t + 2K_{t-1} + 6L_{t-1}$ **(5)**.
(4) in **(5)** $\Rightarrow K_{t+1} = K_t + 2K_{t-1} + 3(K_t - K_{t-1}) \Rightarrow K_{t+1} = 4K_t + K_{t-1} \Rightarrow K_t = 4K_{t-1} + K_{t-2}$ **(6)**.
Karakt. vergelijking van **(6)** is $g^2 - 4g + 1 = 0 \Rightarrow (g-2)^2 - 4 + 1 = 0 \Rightarrow (g-2)^2 = 3 \Rightarrow g = 2 + \sqrt{3} \;\vee\; g = 2 - \sqrt{3}$.
Dus $K_t = A\cdot\left(2+\sqrt{3}\right)^t + B\cdot\left(2-\sqrt{3}\right)^t$.
$K_0 = 40 \Rightarrow A\cdot 1 + B\cdot 1 = A + B = 40$ **(7)**
(1) $\Rightarrow K_1 = 40 + 10 = 50 \Rightarrow A\cdot(2+\sqrt{3}) + B\cdot(2-\sqrt{3}) = 50$ **(8)**
$\begin{cases} A + B = 40 & \textbf{(7)} \;\Big|\; 2+\sqrt{3} \\ (2+\sqrt{3})A + (2-\sqrt{3})B = 50 & \textbf{(8)} \;\Big|\; 1 \end{cases} \Rightarrow \begin{cases} (2+\sqrt{3})A + (2+\sqrt{3})B = 80 + 40\sqrt{3} & \textbf{(9)} \\ (2+\sqrt{3})A + (2-\sqrt{3})B = 50 & \textbf{(8)} \end{cases}$ −
$2\sqrt{3}\cdot B = 30 + 40\sqrt{3} \Rightarrow$
$B = \frac{15}{\sqrt{3}} + 20 = \frac{15}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} + 20 = 20 + 5\sqrt{3}$ in **(7)** $\Rightarrow A + 20 + 5\sqrt{3} = 40 \Rightarrow A = 20 - 5\sqrt{3}$.
Dus $K_t = \left(20 - 5\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right)^t + \left(20 + 5\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right)^t$ **(10)**.

(10) in **(3)** $\Rightarrow \left(20-5\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right)^{t+1}+\left(20+5\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right)^{t+1}=\left(20-5\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right)^{t}+\left(20+5\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right)^{t}+2L_t$

$2L_t=\left(20-5\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right)^{t+1}+\left(20+5\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right)^{t+1}-\left(20-5\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right)^{t}-\left(20+5\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right)^{t}$

$2L_t=\left(20-5\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right)^{t}\cdot\left(2+\sqrt{3}\right)+\left(20+5\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right)^{t}\cdot\left(2-\sqrt{3}\right)-\left(20-5\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right)^{t}-\left(20+5\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right)^{t}$

$2L_t=\left(20-5\sqrt{3}\right)\cdot\left(2+\sqrt{3}-1\right)\cdot\left(2+\sqrt{3}\right)^{t}+\left(20+5\sqrt{3}\right)\cdot\left(2-\sqrt{3}-1\right)\cdot\left(2-\sqrt{3}\right)^{t}$

$2L_t=\left(20-5\sqrt{3}\right)\cdot\left(1+\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right)^{t}+\left(20+5\sqrt{3}\right)\cdot\left(1-\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right)^{t}$

$2L_t=\left(20+20\sqrt{3}-5\sqrt{3}-15\right)\cdot\left(2+\sqrt{3}\right)^{t}+\left(20-20\sqrt{3}+5\sqrt{3}-15\right)\cdot\left(2-\sqrt{3}\right)^{t}$

$2L_t=\left(5+15\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right)^{t}+\left(5-15\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right)^{t}$

Dus $L_t=\left(2\tfrac{1}{2}+7\tfrac{1}{2}\sqrt{3}\right)\cdot\left(2+\sqrt{3}\right)^{t}+\left(2\tfrac{1}{2}-7\tfrac{1}{2}\sqrt{3}\right)\cdot\left(2-\sqrt{3}\right)^{t}$.

57 $x_n=-2x_{n-1}+4y_{n-1}$ **(1)** $\Rightarrow x_{n+1}=-2x_n+4y_n$ **(3)** en $4y_{n-1}=x_n+2x_{n-1}$ **(4)**.
$y_n=-x_{n-1}-2y_{n-1}$ **(2)** substitueren in **(3)** $\Rightarrow x_{n+1}=-2x_n+4\left(-x_{n-1}-2y_{n-1}\right)\Rightarrow x_{n+1}=-2x_n-4x_{n-1}-8y_{n-1}$ **(5)**.
(4) in **(5)** $\Rightarrow x_{n+1}=-2x_n-4x_{n-1}-2\left(x_n+2x_{n-1}\right)\Rightarrow x_{n+1}=-4x_n-8x_{n-1}\Rightarrow x_n=-4x_{n-1}-8x_{n-2}$ **(6)**.
Karakt. vergelijking van **(6)** is $g^2+4g+8=0\Rightarrow(g+2)^2-4+8=0\Rightarrow(g+2)^2=4i^2\Rightarrow g=-2+2i\ \vee\ g=-2-2i$.

Dus $x_n=A\cdot(-2+2i)^n+B\cdot(-2-2i)^n$ *****
$x_0=2\Rightarrow A\cdot1+B\cdot1=A+B=2$ **(7)**
(1) $\Rightarrow x_1=-2\cdot2+4\cdot3=8\Rightarrow A\cdot(-2+2i)+B\cdot(-2-2i)=8$ **(8)**

$$\begin{cases}A+\qquad B=2 & \textbf{(7)}\\(-2+2i)A+(-2-2i)B=8 & \textbf{(8)}\end{cases}\left|\begin{matrix}-2+2i\\1\end{matrix}\right|\Rightarrow\begin{cases}(-2+2i)A+(-2+2i)B=-4+4i & \textbf{(9)}\\(-2+2i)A+(-2-2i)B=\ 8 & \textbf{(8)}\ -\end{cases}$$

$$4i\cdot B=-12+4i\Rightarrow$$

$$B=-\tfrac{3}{i}+1=-\tfrac{3}{i}\cdot\tfrac{i}{i}+1=1+3i \text{ in } \textbf{(7)}\Rightarrow A+1+3i=2\Rightarrow A=1-3i.$$

Dus $x_n=(1-3i)\cdot(-2+2i)^n+(1+3i)\cdot(-2-2i)^n$ **(10)**.

(10) in **(3)** $\Rightarrow(1-3i)\cdot(-2+2i)^{n+1}+(1+3i)\cdot(-2-2i)^{n+1}=-2\left((1-3i)\cdot(-2+2i)^n+(1+3i)\cdot(-2-2i)^n\right)+4y_n$

$4y_n=(1-3i)\cdot(-2+2i)^{n+1}+(1+3i)\cdot(-2-2i)^{n+1}+2(1-3i)\cdot(-2+2i)^n+2(1+3i)\cdot(-2-2i)^n$

$4y_n=(1-3i)\cdot(-2+2i)^n\cdot(-2+2i)+(1+3i)\cdot(-2-2i)^n\cdot(-2-2i)+2(1-3i)\cdot(-2+2i)^n+2(1+3i)\cdot(-2-2i)^n$

$4y_n=(1-3i)\cdot(-2+2i)^n\cdot(-2+2i+2)+(1+3i)\cdot(-2-2i)^n\cdot(-2-2i+2)$

$4y_n=(1-3i)\cdot(-2+2i)^n\cdot2i+(1+3i)\cdot(-2-2i)^n\cdot-2i$

$4y_n=(6+2i)\cdot(-2+2i)^n+(6-2i)\cdot(-2-2i)^n$

$y_n=\left(1\tfrac{1}{2}+\tfrac{1}{2}i\right)\cdot(-2+2i)^n+\left(1\tfrac{1}{2}-\tfrac{1}{2}i\right)\cdot(-2-2i)^n$.

***** OPMERKING:

$|-2+2i|=2\sqrt{2}$ en $\arg(-2+2i)=\tfrac{3}{4}\pi$.

Dus $x_n=\left(A\cdot\cos(\tfrac{3}{4}\pi n)+B\cdot\sin(\tfrac{3}{4}\pi n)\right)\cdot\left(2\sqrt{2}\right)^n$ ■■■

$x_0=2\Rightarrow A\cdot1+B\cdot0=A=2$ **(7*)**

(1) $\Rightarrow x_1=-2\cdot2+4\cdot3=8\Rightarrow A\cdot-\tfrac{1}{2}\sqrt{2}+B\cdot\tfrac{1}{2}\sqrt{2}=8\Rightarrow\sqrt{2}\cdot\left(A\cdot-\tfrac{1}{2}\sqrt{2}+B\cdot\tfrac{1}{2}\sqrt{2}\right)=\sqrt{2}\cdot8\Rightarrow-A+B=8\sqrt{2}$ **(8*)**

(7*) in **(8*)** $\Rightarrow-2+B=8\sqrt{2}\Rightarrow B=2+8\sqrt{2}$.

Dus $x_n=\left(2\cdot\cos(\tfrac{3}{4}\pi n)+(2+8\sqrt{2})\cdot\sin(\tfrac{3}{4}\pi n)\right)\cdot\left(2\sqrt{2}\right)^n$ **(9*)**.

(9*) in **(3)** $\Rightarrow\left(2\cdot\cos(\tfrac{3}{4}\pi(n+1))+(2+8\sqrt{2})\cdot\sin(\tfrac{3}{4}\pi(n+1))\right)\cdot\left(2\sqrt{2}\right)^{n+1}=...+4y_n$

$4y_n=...$

Omschrijven tot de vorm $y_n=\left(C\cdot\cos(\tfrac{3}{4}\pi n)+D\cdot\sin(\tfrac{3}{4}\pi n)\right)\cdot\left(2\sqrt{2}\right)^n$ is zeer bewerkelijk.

Vandaar hier de schrijfwijze $x_n=A\cdot(-2+2i)^n+B\cdot(-2-2i)^n$ uit opgave 57 in plaats van ■■■ hierboven.

Diagnostische toets

D1a $10e^{\frac13\pi i} = 10\left(\cos(\tfrac13\pi) + i\sin(\tfrac13\pi)\right) = 10\left(\tfrac12 + \tfrac12 i\sqrt3\right) = 5 + 5i\sqrt3.$

D1b $6e^{\frac12\pi i} = 6\left(\cos(\tfrac12\pi) + i\sin(\tfrac12\pi)\right) = 6\left(0 + 1\cdot i\right) = 6i.$

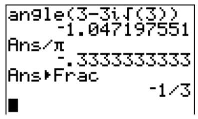
D1c $\dfrac{3}{e^{-\frac16\pi i}} = 3e^{\frac16\pi i} = 3\left(\cos(\tfrac16\pi) + i\sin(\tfrac16\pi)\right) = 3\left(\tfrac12\sqrt3 + \tfrac12 i\right) = \tfrac32\sqrt3 + \tfrac32 i.$

D2a $3 + 3i = 3\sqrt2 e^{\frac14\pi i}.$

D2b $-2 = 2e^{\pi i}.$

D2c $\tfrac12\sqrt2 - \tfrac12 i\sqrt2 = 1e^{-\frac14\pi i} = e^{-\frac14\pi i}.$

D2d $3 - 3i\sqrt3 = 6e^{-\frac13\pi i}.$



D2e $\dfrac{(1+i)^3}{(1-i)^4} = \dfrac{(\sqrt2 e^{\frac14\pi i})^3}{(\sqrt2 e^{-\frac14\pi i})^4} = \dfrac{2\sqrt2 e^{\frac34\pi i}}{4e^{-\pi i}} = \tfrac12\sqrt2 e^{1\frac34\pi i} = \tfrac12\sqrt2 e^{-\frac14\pi i}.$

D2f $\dfrac{1}{(3-i\sqrt3)^4} = \dfrac{1}{(\sqrt{12}\cdot e^{-\frac16\pi i})^4} = \dfrac{1}{144\cdot e^{-\frac23\pi i}} = \dfrac{1}{144}\cdot e^{\frac23\pi i}.$

D3a
$z^2 = -16i = 16e^{-\frac12\pi i + k\cdot 2\pi i}$
$z = 4e^{-\frac14\pi i + k\cdot\pi i}$
$z = 4e^{-\frac14\pi i} \ \vee\ z = 4e^{\frac34\pi i}.$

D3b
$z^3 = 2\sqrt2 - 2i\sqrt2 = 4e^{-\frac14\pi i + k\cdot 2\pi i}$
$z = \sqrt[3]{4}\cdot e^{-\frac1{12}\pi i + k\cdot\frac23\pi i}$
$z = \sqrt[3]{4}\cdot e^{-\frac1{12}\pi i} \ \vee\ z = \sqrt[3]{4}\cdot e^{\frac7{12}\pi i} \ \vee\ z = \sqrt[3]{4}\cdot e^{\frac{15}{12}\pi i}.$

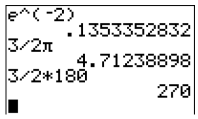
D3c
$(z+2)^2 = -4 = 4i^2$
$z + 2 = 2i \ \vee\ z + 2 = -2i$
$z = -2 + 2i \ \vee\ z = -2 - 2i.$

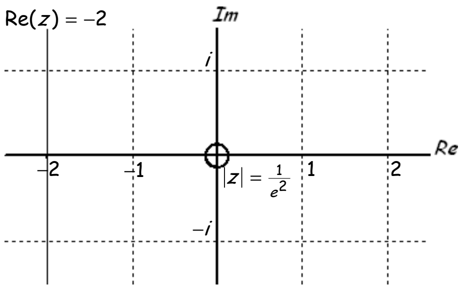
D3d
$z^2 - 4z = -4 + 25i$
$(z-2)^2 - 4 = -4 + 25i$
$(z-2)^2 = 25i = 25e^{\frac12\pi i + k\cdot 2\pi i}$
$z - 2 = 5e^{\frac14\pi i + k\cdot\pi i}$
$z = 2 + 5e^{\frac14\pi i + k\cdot\pi i}$
$z = 2 + 5e^{\frac14\pi i} \ \vee\ z = 2 + 5e^{1\frac14\pi i}$
$z = 2 + 5\left(\cos(\tfrac14\pi) + i\sin(\tfrac14\pi)\right) \ \vee\ z = 2 + 5\left(\cos(1\tfrac14\pi) + i\sin(1\tfrac14\pi)\right)$
$z = 2 + 5\left(\tfrac12\sqrt2 + \tfrac12 i\sqrt2\right) \ \vee\ z = 2 + 5\left(-\tfrac12\sqrt2 - \tfrac12 i\sqrt2\right)$
$z = 2 + \tfrac52\sqrt2 + \tfrac52 i\sqrt2 \ \vee\ z = 2 - \tfrac52\sqrt2 - \tfrac52 i\sqrt2.$

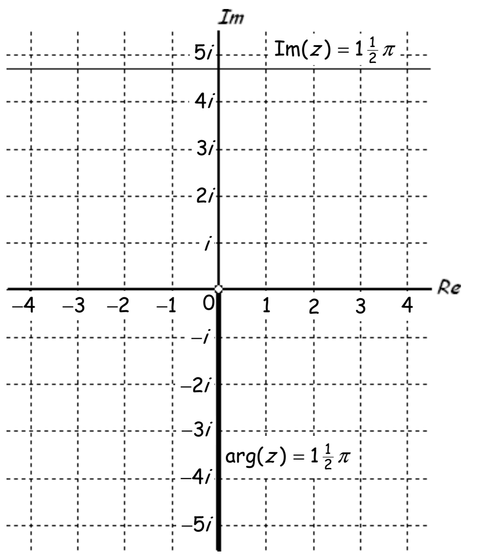
D3e
$(2z - i)^2 = 4i = 4e^{\frac12\pi i + k\cdot 2\pi i}$
$2z - i = 2e^{\frac14\pi i + k\cdot\pi i}$
$2z = i + 2e^{\frac14\pi i + k\cdot\pi i}$
$z = \tfrac12 i + e^{\frac14\pi i + k\cdot\pi i}$
$z = \tfrac12 i + \cos(\tfrac14\pi) + i\sin(\tfrac14\pi) \ \vee\ z = \tfrac12 i + \cos(1\tfrac14\pi) + i\sin(1\tfrac14\pi)$
$z = \tfrac12 i + \tfrac12\sqrt2 + \tfrac12 i\sqrt2 \ \vee\ z = \tfrac12 i - \tfrac12\sqrt2 - \tfrac12 i\sqrt2$
$z = \tfrac12\sqrt2 + \left(\tfrac12 + \tfrac12\sqrt2\right)i \ \vee\ z = -\tfrac12\sqrt2 + \left(\tfrac12 - \tfrac12\sqrt2\right)i.$

D3f
$z^2 - 2z + i\sqrt3 = 0$
$(z-1)^2 - 1 + i\sqrt3 = 0$
$(z-1)^2 = 1 - i\sqrt3 = 2e^{-\frac13\pi i + k\cdot 2\pi i}$
$z - 1 = \sqrt2\cdot e^{-\frac16\pi i + k\cdot\pi i}$
$z = 1 + \sqrt2\cdot e^{-\frac16\pi i + k\cdot\pi i}$
$z = 1 + \sqrt2\cdot e^{-\frac16\pi i} \ \vee\ z = 1 + \sqrt2\cdot e^{\frac56\pi i}$
$z = 1 + \sqrt2\left(\tfrac12\sqrt3 - \tfrac12 i\right) \ \vee\ z = 1 + \sqrt2\left(-\tfrac12\sqrt3 + \tfrac12 i\right)$
$z = 1 + \tfrac12\sqrt6 - \tfrac12 i\sqrt2 \ \vee\ z = 1 - \tfrac12\sqrt6 + \tfrac12 i\sqrt2.$

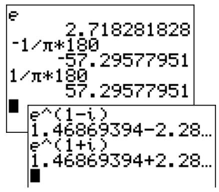
D4a Het beeld van $\text{Re}(z) = -2$ bij $f(z) = e^z$ is de cirkel met middelpunt $z = 0$ en straal e^{-2} ($\approx 0{,}135$), ofwel de cirkel met de vergelijking $|z| = e^{-2} = \frac{1}{e^2}$. (zie hieronder)
Het beeld van $\text{Im}(z) = 1\frac12\pi$ ($\approx 4{,}71$) bij $f(z) = e^z$ is de halve lijn vanaf $z = 0$ die een hoek van $1\frac12\pi$ radialen ($= 270°$) maakt met de postieve reële as, ofwel de halve lijn met vergelijking $\arg(z) = 1\frac12\pi$. (zie hiernaast)

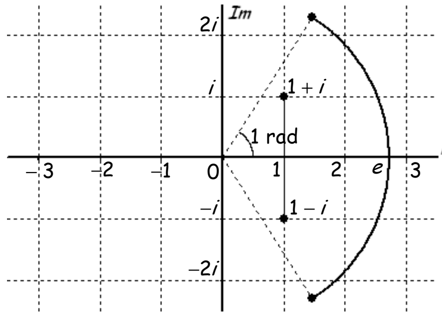






D4b De eindpunten van het lijnstuk zijn $1-i$ en $1+i$, dus $\text{Re}(z)$ is vast. $f(z)=e^z=e^{a+bi}=e^a\cdot e^{bi}$.
Het beeld van het lijnstuk is een deel van een cirkel met middelpunt $z=0$ en straal e^1 ($\approx 2{,}7$) waarbij een hoek wordt doorlopen van -1 tot 1 radialen.





D5a $f(1+i)=\ln(1+i)=\ln(\sqrt2\cdot e^{\frac14\pi i})=\ln(\sqrt2)+\ln(e^{\frac14\pi i})=\ln(\sqrt2)+\frac14\pi i.$

D5b $f(\frac{i}{e})=\ln(\frac{i}{e})=\ln(i)-\ln(e)=\ln(e^{\frac12\pi i})-1=-1+\frac12\pi i.$

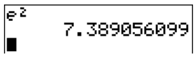
D5c $f(-e^3)=\ln(-e^3)=\ln(-1\cdot e^3)=\ln(-1)+\ln(e^3)=\ln(e^{\pi i})+3=3+\pi i.$

D5d $f(i\sqrt e)=\ln(i\sqrt e)=\ln(i\cdot\sqrt e)=\ln(i)+\ln(\sqrt e)=\ln(e^{\frac12\pi i})+\ln(e^{\frac12})=\frac12\pi i+\frac12=\frac12+\frac12\pi i.$

D5e $f(\frac{2i}{1-i})=\ln(\frac{2i}{1-i})=\ln(2i)-\ln(1-i)=\ln(2\cdot e^{\frac12\pi i})-\ln(\sqrt2\cdot e^{-\frac14\pi i})$
$=\ln(2)+\ln(e^{\frac12\pi i})-\ln(\sqrt2)-\ln(e^{-\frac14\pi i})=\ln(2)+\frac12\pi i-\frac12\ln(2)--\frac14\pi i=\frac12\ln(2)+\frac34\pi i.$

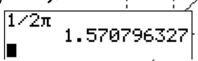
D5f $f(e-ei\sqrt3)=\ln(e-ei\sqrt3)=\ln\left(e\cdot(1-i\sqrt3)\right)=\ln(e)+\ln(1-i\sqrt3)=1+\ln(2e^{-\frac13\pi i})=1+\ln(2)+\ln(2e^{-\frac13\pi i})=1+\ln(2)-\frac13\pi i.$

D6a $|z|=e^2\Rightarrow z=e^2\cdot e^{i\varphi}$ (met $-\pi<\varphi\le\pi$).

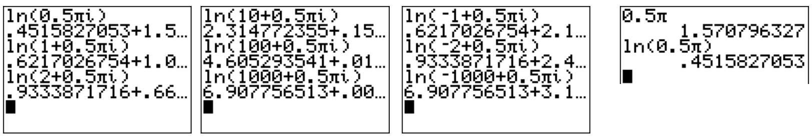


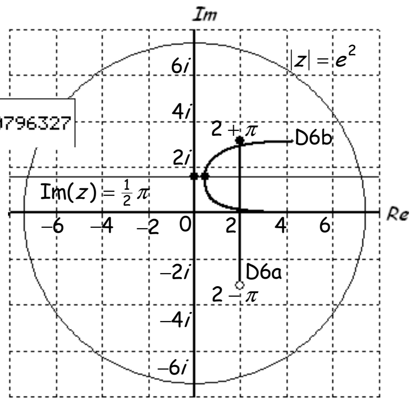
Dus $f(z)=f(e^2\cdot e^{i\varphi})=\ln(e^2\cdot e^{i\varphi})=\ln(e^2)+\ln(e^{i\varphi})=2+i\varphi$ ($-\pi<\varphi\le\pi$).

D6b $\text{Im}(z)=\frac12\pi\Rightarrow z=a+\frac12\pi i$. Dus $f(z)=f(a+\frac12\pi i)=\ln(a+\frac12\pi i)=\ldots$



$f(\frac12\pi\cdot i)=\ln(\frac12\pi\cdot i)=\ln(\frac12\pi)+\ln(i)=\ln(\frac12\pi)+\ln(e^{\frac12\pi i})=\ln(\frac12\pi)+\frac12\pi i.$
Andere punten berekenen met de GR.

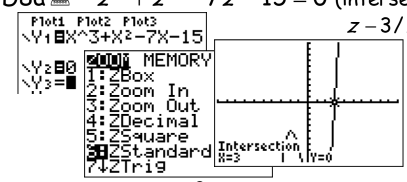




D7a $\cos(\frac12\pi-i)=\frac{e^{i(\frac12\pi-i)}+e^{-i(\frac12\pi-i)}}{2}=\frac{e^{\frac12\pi i+1}+e^{-\frac12\pi-1}}{2}=\frac{e\cdot e^{\frac12\pi i}+e^{-1}\cdot e^{-\frac12\pi}}{2}=\frac{e\cdot(0+1\cdot i)}{2}+\frac{e^{-1}\cdot(0-1\cdot i)}{2}=\left(\frac{e}{2}-\frac{1}{2e}\right)i.$

D7b $\sin(\frac13\pi+2i)=\frac{e^{i(\frac13\pi+2i)}-e^{-i(\frac13\pi+2i)}}{2i}=\frac{e^{\frac13\pi i-2}-e^{-\frac13\pi+2}}{2i}=\frac{e^{-2}\cdot e^{\frac13\pi i}-e^2\cdot e^{-\frac13\pi}}{2i}$
$=\frac{\frac12+\frac12 i\sqrt3}{2ie^2}\cdot\frac{-i}{-i}-\frac{e^2(\frac12-\frac12 i\sqrt3)}{2i}\cdot\frac{-i}{-i}=\frac{\sqrt3}{4e^2}+\frac{e^2\sqrt3}{4}+\left(-\frac{1}{4e^2}+\frac{e^2}{4}\right)i.$

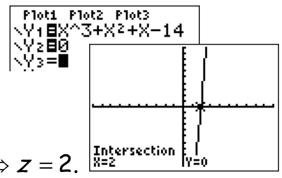
D8a $z^3+z^2-7z-15=0$ (intersect) $\Rightarrow z=3.$



$z-3/z^3+z^2-7z-15\backslash z^2+4z+5$
z^3-3z^2 −
$4z^2-7z-15$
$4z^2-12z$ −
$5z-15$
$5z-15$ −
0

$(z-3)(z^2+4z+5)=0$
$z=3\ \vee\ z^2+4z+5=0$
$z=3\ \vee\ (z+2)^2-4+5=0$
$z=3\ \vee\ (z+2)^2=-1$
$z=3\ \vee\ (z+2)^2=i^2$
$z=3\ \vee\ z+2=-i\ \vee\ z+2=+i$
$z=3\ \vee\ z=-2-i\ \vee\ z=-2+i.$

D8b $(z+1)(z-i)(z+i)=15$
$(z+1)(z^2+1)=15$
$z^3+z+z^2+1=15$
$z^3+z^2+z-14=0$ (intersect) $\Rightarrow z=2.$



$(z-2)(z^2+3z+7)=0$

$z-2/z^3+z^2+z-14\backslash z^2+3z+7$
z^3-2z^2 −
$3z^2+z-14$
$3z^2-6z$ −
$7z-14$
$7z-14$ −
0

$z=2\ \vee\ z^2+3z+7=0$
$z=2\ \vee\ (z+\frac32)^2-\frac94+7=0$
$z=2\ \vee\ (z+\frac32)^2=-\frac{19}{4}$
$z=2\ \vee\ (z+\frac32)^2=\frac{19}{4}i^2$
$z=2\ \vee\ z+\frac32=-\frac12 i\sqrt{19}\ \vee\ z+\frac32=+\frac12 i\sqrt{19}$
$z=2\ \vee\ z=-\frac32-\frac12 i\sqrt{19}\ \vee\ z=-\frac32+\frac12 i\sqrt{19}.$

D9a $z^3+6z-88=0\Rightarrow z^3+6z=88$
$z=u+v$ en $6=-3uv$ geeft $u^3+v^3=88$
Uit $-3uv=6$ volgt $v=-\frac{2}{u}$
$\Rightarrow u^3+\left(-\frac{2}{u}\right)^3=88$

$u^3 + \left(-\frac{2}{u}\right)^3 = 88$

$u^3 - \frac{8}{u^3} = 88$

$u^6 - 88u^3 - 8 = 0$

```
88²-4*1*-8
            7776
2^5*3^5
            7776
■
```

$D = (-88)^2 - 4 \cdot 1 \cdot -8 = 7\,776 \Rightarrow \sqrt{D} = 36\sqrt{6}$

$u^3 = \frac{88 + 36\sqrt{6}}{2} = 44 + 18\sqrt{6}$ is een oplossing

$v^3 = 88 - u^3 = 88 - \left(44 + 18\sqrt{6}\right) = 44 - 18\sqrt{6}$

$z = u + v = \sqrt[3]{44 + 18\sqrt{6}} + \sqrt[3]{44 - 18\sqrt{6}} = 4$

```
3ˣ√(44+18√(6))+3
ˣ√(44-18√(6))
               4
■
```

Nu de staartdeling: (zie hiernaast)

$z - 4 / z^3 \quad + 6z - 88 \setminus z^2 + 4z + 22$

$\underline{z^3 - 4z^2} \; -$

$4z^2 + 6z - 88$

$\underline{4z^2 - 16z} \; -$

$22z - 88$

$\underline{22z - 88} \; -$

0

$z^3 + 6z - 88 = 0$

$(z - 4)(z^2 + 4z + 22) = 0$

$z = 4 \;\vee\; z^2 + 4z + 22 = 0$

$z = 4 \;\vee\; (z + 2)^2 - 4 + 22 = 0$

$z = 4 \;\vee\; (z + 2)^2 = -18$

$z = 4 \;\vee\; (z + 2)^2 = 18i^2$

```
18/2
       9
■
```

$z = 4 \;\vee\; z + 2 = 3i\sqrt{2} \;\vee\; z + 2 = -3i\sqrt{2}$

$\underline{\underline{z = 4 \;\vee\; z = -2 + 3i\sqrt{2} \;\vee\; z = -2 - 3i\sqrt{2}.}}$

D9b $z^3 - 12z = 65$

$z = u + v$ en $-12 = -3uv$ geeft $u^3 + v^3 = 65$

Uit $-3uv = -12$ volgt $v = \frac{4}{u}$ $\Rightarrow$

$u^3 + \left(\frac{4}{u}\right)^3 = 65$

```
4^3
              64
■
```

$u^3 + \frac{64}{u^3} = 65$

$u^6 - 65u^3 + 64 = 0$

$(u^3 - 64)(u^3 - 1) = 0$

$u^3 = 64$ is een oplossing

$v^3 = 65 - u^3 = 65 - 1 = 64$

$z = u + v = \sqrt[3]{64} + \sqrt[3]{1} = 4 + 1 = 5$

De staartdeling: $\quad z - 5 / z^3 \quad - 12z - 65 \setminus z^2 + 5z + 13$

$\underline{z^3 - 5z^2} \; -$

$5z^2 - 12z - 65$

$\underline{5z^2 - 25z} \; -$

$13z - 65$

$\underline{13z - 65} \; -$

0

$z^3 - 12z - 65 = 0$

$(z - 5)(z^2 + 5z + 13) = 0$

$z = 5 \;\vee\; z^2 + 5z + 13 = 0$

$z = 5 \;\vee\; (z + \frac{5}{2})^2 - \frac{25}{4} + 13 = 0$

$z = 5 \;\vee\; (z + \frac{5}{2})^2 = -\frac{27}{4}$

$z = 5 \;\vee\; (z + \frac{5}{2})^2 = \frac{27}{4}i^2$

$z = 5 \;\vee\; z + \frac{5}{2} = \frac{3}{2}i\sqrt{3} \;\vee\; z + \frac{5}{2} = -\frac{3}{2}i\sqrt{3}$

$\underline{\underline{z = 5 \;\vee\; z = -\frac{5}{2} + \frac{3}{2}i\sqrt{3} \;\vee\; z = -\frac{5}{2} - \frac{3}{2}i\sqrt{3}.}}$

D9c $z^3 + 4z^2 + 6z + 4 = 0$

Stel $z = y - \frac{1}{3} \cdot 4 = y - \frac{4}{3}$

$\left(y - \frac{4}{3}\right)^3 + 4\left(y - \frac{4}{3}\right)^2 + 6\left(y - \frac{4}{3}\right) + 4 = 0$

```
16/3-32/3+6▸Frac
             2/3
-64/27+64/9-8+4▸
Frac
           20/27
■
```

$y^3 - \cancel{4y^2} + \frac{16}{3}y - \frac{64}{27} + \cancel{4y^2} - \frac{32}{3}y + \frac{64}{9} + 6y - 8 + 4 = 0$

$y^3 + \frac{2}{3}y = -\frac{20}{27}$

$y = u + v$ en $\frac{2}{3} = -3uv$ geeft $u^3 + v^3 = -\frac{20}{27}$

Uit $-3uv = \frac{2}{3}$ volgt $v = -\frac{2}{9u}$ $\Rightarrow$

$u^3 + \left(-\frac{2}{9u}\right)^3 = -\frac{20}{27}$

$u^3 - \frac{8}{729u^3} = -\frac{20}{27}$

$u^6 + \frac{20}{27}u^3 - \frac{8}{729} = 0$

```
9^3
             729
(20/27)²-4*1*-8/
729▸Frac
           16/27
■
```

$D = \left(\frac{20}{27}\right)^2 - 4 \cdot 1 \cdot -\frac{8}{729} = \frac{16}{27} \Rightarrow \sqrt{D} = \frac{4}{3\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{4}{3}\sqrt{3}$

$u^3 = \frac{-\frac{20}{27} + \frac{4}{9}\sqrt{3}}{2} = -\frac{10}{27} + \frac{2}{9}\sqrt{3}$ is een oplossing

$v^3 = -\frac{20}{27} - u^3 = -\frac{20}{27} - \left(-\frac{10}{27} + \frac{2}{9}\sqrt{3}\right) = -\frac{10}{27} - \frac{2}{9}\sqrt{3}$

$y = u + v = \sqrt[3]{-\frac{10}{27} + \frac{2}{9}\sqrt{3}} + \sqrt[3]{-\frac{10}{27} - \frac{2}{9}\sqrt{3}} = -\frac{2}{3}$

$z = y - \frac{4}{3} = -\frac{2}{3} - \frac{4}{3} = -2$

```
3ˣ√(-10/27+2/9√(
3))+3ˣ√(-10/27-2
/9√(3))▸Frac
            -2/3
■
```

De staartdeling:

$z + 2 / z^3 + 4z^2 + 6z + 4 \setminus z^2 + 2z + 2$

$\underline{z^3 + 2z^2} \; -$

$2z^2 + 6z + 4$

$\underline{2z^2 + 4z} \; -$

$2z + 4$

$\underline{2z + 4} \; -$

0

$z^3 + 4z^2 + 6z + 4 = 0$

$(z + 2)(z^2 + 2z + 2) = 0$

$z = -2 \;\vee\; z^2 + 2z + 2 = 0$

$z = -2 \;\vee\; (z + 1)^2 - 1 + 2 = 0$

$z = -2 \;\vee\; (z + 1)^2 = -1$

$z = -2 \;\vee\; (z + 1)^2 = i^2$

$z = -2 \;\vee\; z + 1 = i \;\vee\; z + 1 = -i$

$\underline{\underline{z = -2 \;\vee\; z = -1 + i \;\vee\; z = -1 - i.}}$

D9d $z^3 + 4z^2 = 24 \Rightarrow z^3 + 4z^2 - 24 = 0$

Stel $z = y - \frac{1}{3} \cdot 4 = y - \frac{4}{3}$

$\left(y - \frac{4}{3}\right)^3 + 4\left(y - \frac{4}{3}\right)^2 - 24 = 0$

```
-64/27+64/9-24▸F
rac
         -520/27
■
```

$y^3 - \cancel{4y^2} + \frac{16}{3}y - \frac{64}{27} + \cancel{4y^2} - \frac{32}{3}y + \frac{64}{9} - 24 = 0$

$y^3 - \frac{16}{3}y = \frac{520}{27}$

$y = u + v$ en $-\frac{16}{3} = -3uv$ geeft $u^3 + v^3 = \frac{520}{27}$

Uit $-3uv = -\frac{16}{3}$ volgt $v = \frac{16}{9u}$ $\Rightarrow$

$u^3 + \left(\frac{16}{9u}\right)^3 = \frac{520}{27}$

```
(16/9)^3▸Frac
        4096/729
■
```

$u^3 + \frac{4096}{729u^3} = \frac{520}{27}$

$u^6 - \frac{520}{27}u^3 + \frac{4096}{729} = 0$

```
(520/27)²-4*1*40
96/729▸Frac
          3136/9
√(Ans▸Frac
            56/3
■
```

$D = \left(-\frac{520}{27}\right)^2 - 4 \cdot 1 \cdot \frac{4096}{729} = \frac{3136}{9} \Rightarrow \sqrt{D} = \frac{56}{3}$

$u^3 = \frac{\frac{520}{27} + \frac{56}{3}}{2} = \frac{512}{27}$ is een oplossing

```
(520/27+56/3)/2▸
Frac
          512/27
■
```

$v^3 = \frac{520}{27} - u^3 = \frac{520}{27} - \frac{512}{27} = \frac{8}{27}$

$y = u + v = \sqrt[3]{\frac{512}{27}} + \sqrt[3]{\frac{8}{27}} = \frac{10}{3}$

```
3ˣ√(512/27)+3ˣ√(
8/27)▸Frac
            10/3
■
```

$z = y - \frac{4}{3} = \frac{10}{3} - \frac{4}{3} = 2$

De staartdeling:

$z + 2 / z^3 + 4z^2 + 6z + 4 \setminus z^2 + 2z + 2$

$\underline{z^3 + 2z^2} \; -$

$2z^2 + 6z + 4$

$\underline{2z^2 + 4z} \; -$

$2z + 4$

$\underline{2z + 4} \; -$

0

$z^3 + 4z^2 - 24 = 0$

$(z - 2)(z^2 + 6z + 12) = 0$

$z = 2 \;\vee\; z^2 + 6z + 12 = 0$

$z = 2 \;\vee\; (z + 3)^2 - 9 + 12 = 0$

$z = 2 \;\vee\; (z + 3)^2 = -3$

$z = 2 \;\vee\; (z + 3)^2 = 3i^2$

$z = 2 \;\vee\; z + 3 = i\sqrt{3} \;\vee\; z + 3 = -i\sqrt{3}$

$\underline{\underline{z = 2 \;\vee\; z = -3 + i\sqrt{3} \;\vee\; z = -3 - i\sqrt{3}.}}$

D10a Substitueer $u_n = g^n$ in $u_n = 4u_{n-1} - 3u_{n-2}$.
$g^2 - 4g + 3 = 0$
$(g-1)(g-3) = 0$
$g = 1 \;\vee\; g = 3.$
Dus $u_n = A \cdot 1^n + B \cdot 3^n = A + B \cdot 3^n$.
$u_0 = 3 \Rightarrow A + B \cdot 1 = A + B = 3$ **(1)**
$u_1 = -1 \Rightarrow A + B \cdot 3 = A + 3B = -1$ **(2)**
$\begin{cases} A + B = 3 & \textbf{(1)} \\ A + 3B = -1 & \textbf{(2)} \end{cases} \; -$
$-2B = 4 \Rightarrow B = -2$ **(3)**
(3) in **(1)** $\Rightarrow A - 2 = 3 \Rightarrow A = 5.$
Dus $u_n = 5 - 2 \cdot 3^n$.

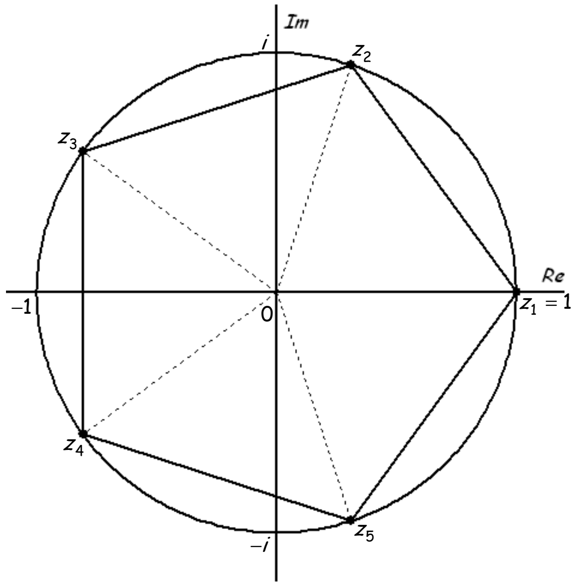
D10b Substitueer $u_n = g^n$ in $u_n = 9u_{n-2}$.
$g^2 - 9 = 0$
$(g-3)(g+3) = 0$
$g = 3 \;\vee\; g = -3.$
Dus $u_n = A \cdot 3^n + B \cdot (-3)^n$.
$u_0 = 5 \Rightarrow A \cdot 1 + B \cdot 1 = A + B = 5$ **(1)**
$u_1 = -3 \Rightarrow A \cdot 3 + B \cdot -3 = 3A - 3B = -3$ **(2)**
$\begin{cases} A + B = 5 & \textbf{(3)} \\ A - B = -1 & \textbf{(2*)} \end{cases} \; +$
$2A = 4 \Rightarrow A = 2$ **(4)**
(4) in **(1)** $\Rightarrow 2 + B = 5 \Rightarrow B = 3.$
Dus $u_n = 2 \cdot 3^n + 3 \cdot (-3)^n$.

D10c Substitueer $u_n = g^n$ in $u_n = -2u_{n-1} - 2u_{n-2}$.
$g^2 + 2g + 2 = 0$
$(g+1)^2 - 1 + 2 = 0$
$(g+1)^2 = -1$
$(g+1)^2 = i^2$
$g = -1 + i \;\vee\; g = -1 - i.$
$|-1 + i| = \sqrt{2}$ en $\arg(-1+i) = \frac{3}{4}\pi$.
Dus $u_n = \left(A\cos(\frac{3}{4}\pi n) + B\sin(\frac{3}{4}\pi n)\right) \cdot \sqrt{2}^n$.
$u_0 = 2\sqrt{2} \Rightarrow (A \cdot 1 + B \cdot 0) \cdot 1 = A = 2\sqrt{2}$ **(1)**
$u_1 = 1 \Rightarrow \left(A \cdot -\frac{1}{2}\sqrt{2} + B \cdot \frac{1}{2}\sqrt{2}\right) \cdot \sqrt{2} = -A + B = 1$ **(2)**
(1) in **(2)** $\Rightarrow -2\sqrt{2} + B = 1 \Rightarrow B = 1 + 2\sqrt{2}.$
Dus $u_n = \left(2\sqrt{2}\cos(\frac{3}{4}\pi n) + (1 + 2\sqrt{2})\sin(\frac{3}{4}\pi n)\right) \cdot \sqrt{2}^n$.

D10d Substitueer $u_n = g^n$ in $u_n = 6u_{n-1} - 9u_{n-2}$.
$g^2 - 6g + 9 = 0$
$(g-3)(g-3) = 0$
$g = 3 \;\vee\; g = 3.$
Dus $u_n = (A + Bn) \cdot 3^n$.
$u_0 = 5 \Rightarrow A + B \cdot 0 = A = 5$ **(1)**
$u_1 = 18 \Rightarrow (A + B \cdot 1) \cdot 3 = 18 \Rightarrow A + B = 6$ **(2)**
(1) in **(2)** $\Rightarrow 5 + B = 6 \Rightarrow B = 1.$
Dus $u_n = (5 + n) \cdot 3^n$.

D11 $x_n = 3x_{n-1} - 5y_{n-1}$ **(1)** $\Rightarrow x_{n+1} = 3x_n - 5y_n$ **(3)** en $5y_{n-1} = -x_n + 3x_{n-1}$ **(4)**.
$y_n = x_{n-1} + y_{n-1}$ **(2)** substitueren in **(3)** $\Rightarrow x_{n+1} = 3x_n - 5(x_{n-1} + y_{n-1}) \Rightarrow x_{n+1} = 3x_n - 5x_{n-1} - 5y_{n-1}$ **(5)**.
(4) in **(5)** $\Rightarrow x_{n+1} = 3x_n - 5x_{n-1} - 1 \cdot (-x_n + 3x_{n-1}) \Rightarrow x_{n+1} = 4x_n - 8x_{n-1} \Rightarrow x_n = 4x_{n-1} - 8x_{n-2}$ **(6)**.
Karakt. vergelijking van **(6)** is $g^2 - 4g + 8 = 0 \Rightarrow (g-2)^2 - 4 + 8 = 0 \Rightarrow (g-2)^2 = 4i^2 \Rightarrow g = 2 + 2i \;\vee\; g = 2 - 2i$.
Dus $x_n = A \cdot (2+2i)^n + B \cdot (2-2i)^n$
$x_0 = 1 \Rightarrow A \cdot 1 + B \cdot 1 = A + B = 1$ **(7)**
(1) $\Rightarrow x_1 = 3 \cdot 1 - 5 \cdot 2 = -7 \Rightarrow A \cdot (2+2i) + B \cdot (2-2i) = -7$ **(8)**
$\begin{cases} A + B = 1 & \textbf{(7)} \\ (2+2i)A + (2-2i)B = -7 & \textbf{(8)} \end{cases} \left| \begin{matrix} 2+2i \\ 1 \end{matrix} \right| \Rightarrow \begin{cases} (2+2i)A + (2+2i)B = 2 + 2i & \textbf{(9)} \\ (2+2i)A + (2-2i)B = -7 & \textbf{(8)} \end{cases} \; -$
$4iB = 9 + 2i \Rightarrow$
$B = \frac{9}{4i} + \frac{1}{2} = \frac{9}{4i} \cdot \frac{i}{i} + \frac{1}{2} = \frac{1}{2} - \frac{9}{4}i$ in **(7)** $\Rightarrow A + \frac{1}{2} - \frac{9}{4}i = 1 \Rightarrow A = \frac{1}{2} + \frac{9}{4}i.$
Dus $x_n = \left(\frac{1}{2} + \frac{9}{4}i\right) \cdot (2+2i)^n + \left(\frac{1}{2} - \frac{9}{4}i\right) \cdot (2-2i)^n$ **(10)**.
(10) in **(3)** $\Rightarrow \left(\frac{1}{2} + \frac{9}{4}i\right) \cdot (2+2i)^{n+1} + \left(\frac{1}{2} - \frac{9}{4}i\right) \cdot (2-2i)^{n+1} = 3\left(\left(\frac{1}{2} + \frac{9}{4}i\right) \cdot (2+2i)^n + \left(\frac{1}{2} - \frac{9}{4}i\right) \cdot (2-2i)^n\right) - 5y_n$
$5y_n = -\left(\frac{1}{2} + \frac{9}{4}i\right) \cdot (2+2i)^n \cdot (2+2i) - \left(\frac{1}{2} - \frac{9}{4}i\right) \cdot (2-2i)^n \cdot (2-2i) + 3\left(\frac{1}{2} + \frac{9}{4}i\right) \cdot (2+2i)^n + 3\left(\frac{1}{2} - \frac{9}{4}i\right) \cdot (2-2i)^n$
$5y_n = \left(\frac{1}{2} + \frac{9}{4}i\right) \cdot (2+2i)^n \cdot (-2 - 2i + 3) + \left(\frac{1}{2} - \frac{9}{4}i\right) \cdot (2-2i)^n \cdot (-2 + 2i + 3)$
$5y_n = \left(\frac{1}{2} + \frac{9}{4}i\right) \cdot (2+2i)^n \cdot (1 - 2i) + \left(\frac{1}{2} - \frac{9}{4}i\right) \cdot (2-2i)^n \cdot (1 + 2i)$
$5y_n = \left(\frac{1}{2} + \frac{9}{4}i\right) \cdot (2+2i)^n \cdot (1 - 2i) + \left(\frac{1}{2} - \frac{9}{4}i\right) \cdot (2-2i)^n \cdot (1 + 2i)$
$5y_n = \left(\frac{1}{2} - i + \frac{9}{4}i + \frac{9}{2}\right) \cdot (2+2i)^n + \left(\frac{1}{2} + i - \frac{9}{4}i + \frac{9}{2}\right) \cdot (2-2i)^n$
$5y_n = \left(5 + \frac{5}{4}i\right) \cdot (2+2i)^n + \left(5 - \frac{5}{4}i\right) \cdot (2-2i)^n$
$y_n = \left(1 + \frac{1}{4}i\right) \cdot (2+2i)^n + \left(1 - \frac{1}{4}i\right) \cdot (2-2i)^n$.

Gemengde opgaven 12. Complexe getallen gebruiken



G32a $z^5 = 1 = e^{k \cdot 2\pi i}$

$z = e^{k \cdot \frac{2}{5}\pi i}$

$z_1 = 1 \;\vee\; z_2 = e^{\frac{2}{5}\pi i} \;\vee\; z_3 = e^{\frac{4}{5}\pi i} \;\vee\; z_4 = e^{\frac{6}{5}\pi i} \;\vee\; z_5 = e^{\frac{8}{5}\pi i}.$

G32b $z_1 = 1 = e^0 = \cos(0) + i\sin(0)$

$z_2 = e^{\frac{2}{5}\pi i} = \cos(\frac{2}{5}\pi) + i\sin(\frac{2}{5}\pi)$

$z_3 = e^{\frac{4}{5}\pi i} = \cos(\frac{4}{5}\pi) + i\sin(\frac{4}{5}\pi)$

$z_4 = e^{\frac{6}{5}\pi i} = \cos(\frac{6}{5}\pi) + i\sin(\frac{6}{5}\pi)$

$z_5 = e^{\frac{8}{5}\pi i} = \cos(\frac{8}{5}\pi) + i\sin(\frac{8}{5}\pi)$

Hieruit volgt: $\sin(0) + \sin(\frac{2}{5}\pi) + \sin(\frac{4}{5}\pi) + \sin(\frac{6}{5}\pi) + \sin(\frac{8}{5}\pi) = 0.$

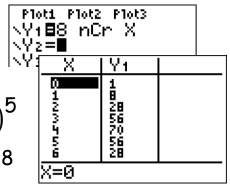
$\sin(0) = 0 \Rightarrow \sin(\frac{2}{5}\pi) + \sin(\frac{4}{5}\pi) + \sin(\frac{6}{5}\pi) + \sin(\frac{8}{5}\pi) = 0.$

Ook geldt: $\cos(0) + \cos(\frac{2}{5}\pi) + \cos(\frac{4}{5}\pi) + \cos(\frac{6}{5}\pi) + \cos(\frac{8}{5}\pi) = 0.$

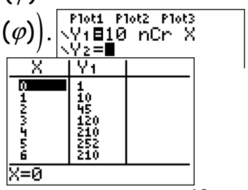
$\cos(0) = 1 \Rightarrow \cos(\frac{2}{5}\pi) + \cos(\frac{4}{5}\pi) + \cos(\frac{6}{5}\pi) + \cos(\frac{8}{5}\pi) = -1.$

G32c $\sum_{k=1}^{10}\left(\sin(\frac{2k\pi}{10}) + \cos(\frac{2k\pi}{10})\right) = \sum_{k=1}^{10}\sin(\frac{2k\pi}{10}) + \sum_{k=1}^{10}\cos(\frac{2k\pi}{10}) = 0 + 0 = 0.$

G33a $(\cos(\varphi) + i\sin(\varphi))^8 = \cos(8\varphi) + i\sin(8\varphi)$, dus



$$\cos(8\varphi) + i\sin(8\varphi) = \binom{8}{0}\cdot\cos^8(\varphi)\cdot(i\sin(\varphi))^0 + \binom{8}{1}\cdot\cos^7(\varphi)\cdot(i\sin(\varphi))^1 + \binom{8}{2}\cdot\cos^6(\varphi)\cdot(i\sin(\varphi))^2$$
$$+\binom{8}{3}\cdot\cos^5(\varphi)\cdot(i\sin(\varphi))^3 + \binom{8}{4}\cdot\cos^4(\varphi)\cdot(i\sin(\varphi))^4 + \binom{8}{5}\cdot\cos^3(\varphi)\cdot(i\sin(\varphi))^5$$
$$+\binom{8}{6}\cdot\cos^2(\varphi)\cdot(i\sin(\varphi))^6 + \binom{8}{7}\cdot\cos^1(\varphi)\cdot(i\sin(\varphi))^7 + \binom{8}{8}\cdot\cos^0(\varphi)\cdot(i\sin(\varphi))^8$$
$$= \cos^8(\varphi) + 8i\cos^7(\varphi)\sin(\varphi) - 28\cos^6(\varphi)\sin^2(\varphi) - 56i\cos^5(\varphi)\sin^3(\varphi) + 70\cos^4(\varphi)\sin^4(\varphi)$$
$$+56i\cos^3(\varphi)\sin^5(\varphi) - 28\cos^2(\varphi)\sin^6(\varphi) - 8i\cos(\varphi)\sin^7(\varphi) + \sin^8(\varphi)$$
$$= \cos^8(\varphi) - 28\cdot\cos^6(\varphi)\sin^2(\varphi) + 70\cos^4(\varphi)\sin^4(\varphi) - 28\cos^2(\varphi)\sin^6(\varphi) + \sin^8(\varphi)$$
$$+i\cdot\left(8i\cos^7(\varphi)\sin(\varphi) - 56\cos^5(\varphi)\sin^3(\varphi) + 56\cos^3(\varphi)\sin^5(\varphi) - 8\cos(\varphi)\sin^7(\varphi)\right).$$



Dus $\cos(8\varphi) = \cos^8(\varphi) - 28\cdot\cos^6(\varphi)\sin^2(\varphi) + 70\cos^4(\varphi)\sin^4(\varphi) - 28\cos^2(\varphi)\sin^6(\varphi) + \sin^8(\varphi).$

G33b Zie G33a: $\sin(8\varphi) = 8i\cos^7(\varphi)\sin(\varphi) - 56\cos^5(\varphi)\sin^3(\varphi) + 56\cos^3(\varphi)\sin^5(\varphi) - 8\cos(\varphi)\sin^7(\varphi).$

G33c $(\cos(\varphi) + i\sin(\varphi))^8 = \cos(10\varphi) + i\sin(10\varphi)$

$$= \binom{10}{0}\cdot\cos^{10}(\varphi)\cdot(i\sin(\varphi))^0 + \binom{10}{1}\cdot\cos^9(\varphi)\cdot(i\sin(\varphi))^1 + \ldots + \binom{10}{9}\cdot\cos^1(\varphi)\cdot(i\sin(\varphi))^9 + \binom{10}{10}\cdot\cos^0(\varphi)\cdot(i\sin(\varphi))^{10}.$$

$\cos(10\varphi) = \cos^{10}(\varphi) - 45\cdot\cos^8(\varphi)\sin^2(\varphi) + 210\cos^6(\varphi)\sin^4(\varphi) - 210\cos^4(\varphi)\sin^6(\varphi) + 45\cos^4(\varphi)\sin^8(\varphi) - \sin^{10}(\varphi).$

G34a $z^4 + 4z^2 + 3 = 0$

$(z^2 + 1)(z^2 + 3) = 0$

$z^2 = -1 \;\vee\; z^2 = -3$

$z^2 = i^2 \;\vee\; z^2 = 3i^2$

$z = i \;\vee\; z = -i \;\vee\; z = i\sqrt{3} \;\vee\; z = -i\sqrt{3}.$

G34b $z^4 + 4z^3 + 5z^2 = 0$

$z^2(z^2 + 4z + 5) = 0$

$z^2 = 0 \;\vee\; z^2 + 4z + 5 = 0$

$z = 0 \;\vee\; (z+2)^2 - 4 + 5 = 0$

$z = 0 \;\vee\; (z+2)^2 = -1$

$z = 0 \;\vee\; (z+2)^2 = i^2$

$z = 0 \;\vee\; z + 2 = i \;\vee\; z + 2 = -i$

$z = 0 \;\vee\; z - 2 + i \;\vee\; z = -2 - i.$

G34c $z^6 - 7z^3 - 8 = 0$

$(z^3 + 1)(z^3 - 8) = 0$

$z^3 = -1 \;\vee\; z^3 = 8$

$z^3 = e^{\pi i + k\cdot 2\pi i} \;\vee\; z^3 = 2e^{k\cdot 2\pi i}$

$z = e^{\frac{1}{3}\pi i + k\cdot\frac{2}{3}\pi i} \;\vee\; z = 2e^{k\cdot\frac{2}{3}\pi i}$

$z = e^{\frac{1}{3}\pi i} \;\vee\; z = -1 \;\vee\; z = e^{\frac{5}{3}\pi i} \;\vee\; z = 2 \;\vee\; z = 2e^{\frac{2}{3}\pi i} \;\vee\; z = 2e^{\frac{4}{3}\pi i}.$

G34d $z^3 - 9z = 80$

$z = u + v$ en $-9 = -3uv$ geeft $u^3 + v^3 = 80$
Uit $-3uv = -9$ volgt $v = \frac{3}{u}$ $\Rightarrow$

$u^3 + \left(\frac{3}{u}\right)^3 = 80$

$u^3 + \frac{27}{u^3} = 80$

$u^6 - 80u^3 + 27 = 0$

```
80²-4*1*27
                6292
Ans/2/2/11/11
                  13
```

$D = (-80)^2 - 4 \cdot 1 \cdot 27 = 6292 \Rightarrow \sqrt{D} = 22\sqrt{13}$

$u^3 = \frac{80 + 22\sqrt{13}}{2} = 40 + 11\sqrt{13}$

$v^3 = 80 - u^3 = 80 - \left(40 + 11\sqrt{13}\right) = 40 - 11\sqrt{13}$

$z = u + v = \sqrt[3]{40 + 11\sqrt{13}} + \sqrt[3]{40 - 11\sqrt{13}} = 5$

```
3ˣ√(40+11√(13))+
3ˣ√(40-11√(13))
                   5
```

De staartdeling: $z - 5 / z^3 \quad - 9z - 80 \backslash z^2 + 5z + 16$
$z^3 - 5z^2$ −
$5z^2 - 9z - 80$
$5z^2 - 25z$ −
$16z - 80$
$16z - 80$ −
0

$z^3 - 9z - 80 = 0$

$(z - 5)(z^2 + 5z + 16) = 0$

$z = 5 \ \vee \ z^2 + 5z + 16 = 0$

$z = 5 \ \vee \ (z + \frac{5}{2})^2 - \frac{25}{4} + 16 = 0$

```
-25/4+16
              9.75
Ans▸Frac
              39/4
```

$z = 5 \ \vee \ (z + \frac{5}{2})^2 = -\frac{39}{4}$

$z = 5 \ \vee \ (z + \frac{5}{2})^2 = \frac{39}{4} i^2$

$z = 5 \ \vee \ z + \frac{5}{2} = \frac{1}{2} i\sqrt{39} \ \vee \ z + \frac{5}{2} = -\frac{1}{2} i\sqrt{39}$

$z = 5 \ \vee \ z = -\frac{5}{2} + \frac{1}{2} i\sqrt{39} \ \vee \ z = -\frac{5}{2} - \frac{1}{2} i\sqrt{39}.$

G34e $z^3 - 6z^2 + 18z - 40 = 0$

Stel $z = y - \frac{1}{3} \cdot -6 = y + 2$

$(y + 2)^3 - 6(y + 2)^2 + 18(y + 2) - 40 = 0$

$y^3 + 6y^2 + 12y + 8 - 6y^2 - 24y - 24 + 18y + 36 - 40 = 0$

$y^3 + 6y = 20$

$y = u + v$ en $6 = -3uv$ geeft $u^3 + v^3 = 20$
Uit $-3uv = 6$ volgt $v = -\frac{2}{u}$ $\Rightarrow$

$u^3 + \left(-\frac{2}{u}\right)^3 = 20$

$u^3 - \frac{8}{u^3} = 20$

$u^6 - 20u^3 - 8 = 0$

```
20²-4*1*-8
                 432
Ans/2/2/2/2/3/3
                   3
```

$D = (-20)^2 - 4 \cdot 1 \cdot -8 = 432 \Rightarrow \sqrt{D} = 12\sqrt{3}$

$u^3 = \frac{20 + 12\sqrt{3}}{2} = 10 + 6\sqrt{3}$ is een oplossing

$v^3 = 20 - u^3 = 20 - \left(10 + 6\sqrt{3}\right) = 10 - 6\sqrt{3}$

$y = u + v = \sqrt[3]{10 + 6\sqrt{3}} + \sqrt[3]{10 - 6\sqrt{3}} = 2$

```
3ˣ√(10+6√(3))+3ˣ
√(10-6√(3))
                   2
```

$z = y + 2 = 2 + 2 = 4$

De staartdeling: $z - 4 / z^3 - 6z^2 + 18z - 40 \backslash z^2 - 2z + 10$
$z^3 - 4z^2$ −
$-2z^2 + 18z - 40$
$-2z^2 + 8z$ −
$10z - 40$
$10z - 40$ −
0

$z^3 - 6z^2 + 18z - 40 = 0$

$(z - 4)(z^2 - 2z + 10) = 0$

$z = 4 \ \vee \ z^2 - 2z + 10 = 0$

$z = 4 \ \vee \ (z + 1)^2 - 1 + 10 = 0$

$z = 4 \ \vee \ (z + 1)^2 = -9$

$z = 4 \ \vee \ (z + 1)^2 = 9i^2$

$z = 4 \ \vee \ z + 1 = 3i \ \vee \ z + 1 = -3i$

$z = 4 \ \vee \ z = -1 + 3i \ \vee \ z = -1 - 3i.$

G35a $\text{Re}(z) = 2 \Rightarrow z = 2 + bi.$

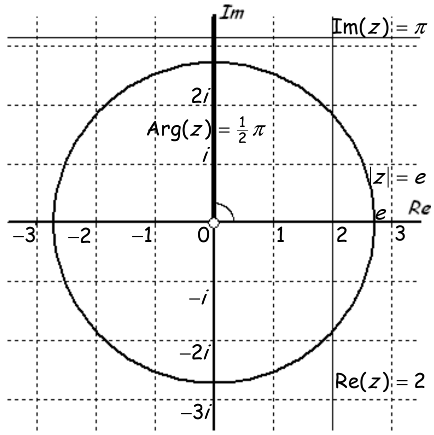
$f(2 + bi) = e^{\frac{1}{2} \cdot (2 + bi)} = e^{1 + \frac{1}{2}bi} = e \cdot e^{\frac{1}{2}bi}.$

Dus het beeld van $\text{Re}(z) = 2$ is de cirkel met middelpunt $z = 0$ en straal e, ofwel de cirkel met vergelijking $|z| = e$.

$\text{Im}(z) = \pi \Rightarrow z = a + \pi i.$

$f(a + \pi i) = e^{\frac{1}{2} \cdot (a + \pi i)} = e^{\frac{1}{2}a + \frac{1}{2}\pi i} = e^{\frac{1}{2}a} \cdot e^{\frac{1}{2}\pi i}.$

Dus het beeld van $\text{Im}(z) = \pi$ is de halve lijn met vanaf $z = 0$ die een hoek van $\frac{1}{2}\pi$ radialen maakt met de positieve reële as, ofwel de halve lijn met vergelijking $Arg(z) = \frac{1}{2}\pi$.

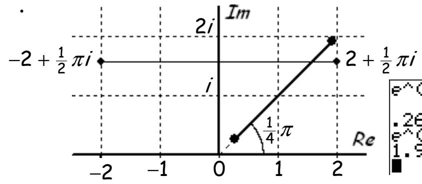


G35b De eindpunten van het lijnstuk zijn $z = -2 + \frac{1}{2}\pi i$ en $z = 2 + \frac{1}{2}\pi i$, dus $\text{Im}(z) = \frac{1}{2}\pi$ is vast.

$z = -2 + \frac{1}{2}\pi i \Rightarrow f(-2 + \frac{1}{2}\pi i) = e^{\frac{1}{2} \cdot (-2 + \frac{1}{2}\pi i)} = e^{-1 + \frac{1}{4}\pi i} = e^{-1} \cdot e^{\frac{1}{4}\pi i}.$

$z = 2 + \frac{1}{2}\pi i \Rightarrow f(2 + \frac{1}{2}\pi i) = e^{\frac{1}{2} \cdot (2 + \frac{1}{2}\pi i)} = e^{1 + \frac{1}{4}\pi i} = e^{1} \cdot e^{\frac{1}{4}\pi i}.$

Dus het beeld is het lijnstuk onder een hoek van $\frac{1}{4}\pi$ radialen maakt met de positieve reële as tussen de punten die een afstand tot $z = 0$ hebben van respectievelijk $e^{-1} = \frac{1}{e}$ en e.




```
e^(1/2(-2+1/2πi)
.2601300475+.26…
e^(1/2(2+1/2πi)
1.922115514+1.9…
```

G36a $i = 1 \cdot e^{\frac{1}{2}\pi i}$ en op de eenheidscirkel is $z = 1 \cdot e^{i\phi}$.

$f(z) = \ln(iz) = \ln(e^{\frac{1}{2}\pi i} \cdot e^{i\phi}) = \ln(e^{\frac{1}{2}\pi i}) + \ln(e^{i\phi}) = \frac{1}{2}\pi i + i\phi = (\frac{1}{2}\pi + \phi)i.$

Maar in dit hoofdstuk beperken we ons tot $-\pi < \text{Im}(f(z)) \leq \pi.$

(zie bovenaan blz. 145, dus zo gauw $\text{Im}(f(z)) > \pi$ trekken we er 2π vanaf)

Dus het beeld van de eenheidscirkel is het lijnstuk met eindpunten $z = -\pi i$ en $z = \pi i$.

```
π
        3.141592654
```

De figuur staat op de volgende bladzijde.

G36b $\text{Re}(z)=\frac{1}{2}\pi$, dus $z=\frac{1}{2}\pi+bi$ en $iz=i\cdot(\frac{1}{2}\pi+bi)=-b+\frac{1}{2}\pi i$.

$f\left(\frac{1}{2}\pi+bi\right)=\ln\left(i\cdot(\frac{1}{2}\pi+bi)\right)$

$=\ln\left(-b+\frac{1}{2}\pi i\right)$

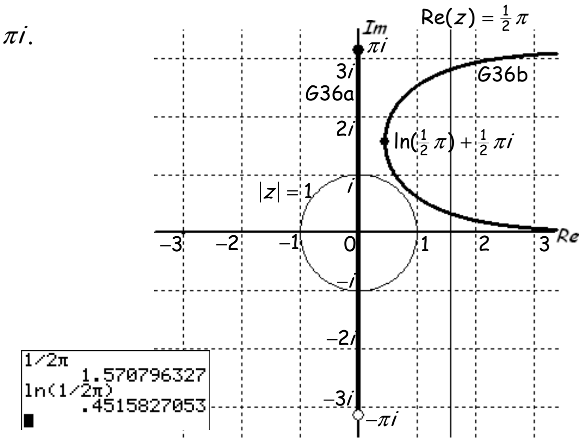
$=\ln\left(\sqrt{(-b)^2+(\frac{1}{2}\pi)^2}\cdot e^{i\phi}\right)$

$(b=0\Rightarrow\phi=\frac{1}{2}\pi$ en $-b\to\infty$ dan $\phi\to 0$ en $-b\to-\infty$ dan $\phi\to\pi)$

$=\ln\left(\sqrt{(-b)^2+(\frac{1}{2}\pi)^2}\right)+\ln\left(e^{i\phi}\right)$

$=\ln\left(\sqrt{(-b)^2+(\frac{1}{2}\pi)^2}\right)+i\phi$ met $-\frac{1}{2}\pi<\phi<\frac{1}{2}\pi$.

$f\left(\frac{1}{2}\pi+0i\right)=\ln\left(\sqrt{0^2+(\frac{1}{2}\pi)^2}\right)+i\cdot\frac{1}{2}\pi=\ln\left(\frac{1}{2}\pi\right)+\frac{1}{2}\pi i$.



G37a $\cos(i-\frac{1}{2}\pi)=\dfrac{e^{i\cdot(i-\frac{1}{2}\pi)}+e^{-i\cdot(i-\frac{1}{2}\pi)}}{2}=\dfrac{e^{-1-\frac{1}{2}\pi i}+e^{1+\frac{1}{2}\pi i}}{2}=\dfrac{e^{-1}\cdot e^{-\frac{1}{2}\pi i}+e\cdot e^{\frac{1}{2}\pi}}{2}=\dfrac{-e^{-1}\cdot i+ei}{2}\cdot\dfrac{i}{i}=\dfrac{e^{-1}-e}{2i}=\dfrac{e^{i\cdot i}-e^{-i\cdot i}}{2i}=\sin(i)$.

G37b $\sin(2i)=\dfrac{e^{2i\cdot i}+e^{-2i\cdot i}}{2i}=\dfrac{e^{-2}+e^{2}}{2i}=\dfrac{(e^{-1}+e)\cdot(e^{-1}-e)}{2i}=2\cdot\dfrac{e^{-1}+e}{2}\cdot\dfrac{e^{-1}-e}{2i}=2\cdot\dfrac{e^{i\cdot i}+e^{-i\cdot i}}{2}\cdot\dfrac{e^{i\cdot i}-e^{-i\cdot i}}{2i}=2\cos(i)\sin(i)$.

G38a Substitueer $u_n=g^n$ in $u_n=-4u_{n-1}+5u_{n-2}$.

$g^2+4g-5=0$

$(g-1)(g+5)=0$

$g=1 \;\vee\; g=-5$.

Dus $u_n=A\cdot 1^n+B\cdot(-5)^n=A+B\cdot(-5)^n$.

$u_0=1\Rightarrow A+B\cdot 1=A+B=1$ **(1)**

$u_1=5\Rightarrow A+B\cdot -5=A-5B=5$ **(2)**

$\begin{cases}A+B=1 & \textbf{(1)}\\ A-5B=5 & \textbf{(2)}\end{cases}$ −

$6B=-4\Rightarrow B=-\frac{4}{6}=-\frac{2}{3}$ **(3)**

(3) in **(1)** $\Rightarrow A-\frac{2}{3}=1\Rightarrow A=1\frac{2}{3}$.

Dus $u_n=1\frac{2}{3}-\frac{2}{3}\cdot(-5)^n$.

G38b Substitueer $u_n=g^n$ in $u_n=6u_{n-1}-9u_{n-2}$.

$g^2-6g+9=0$

$(g-3)(g-3)=0$

$g=3 \;\vee\; g=3$.

Dus $u_n=(A+Bn)\cdot 3^n$.

$u_0=6\Rightarrow(A+B\cdot 0)\cdot 1=A=6$ **(1)**

$u_1=8\Rightarrow(A+B\cdot 2)\cdot 3=3A+3B=8\Rightarrow A+B=\frac{8}{3}$ **(2)**

(1) in **(2)** $\Rightarrow 6+B=\frac{8}{3}\Rightarrow B=-\frac{10}{3}$.

Dus $u_n=(6-\frac{10}{3}n)\cdot 3^n$.

G38c Substitueer $u_n=g^n$ in $u_n=8u_{n-1}-32u_{n-2}$.

$g^2-8g+32=0$

$(g-4)^2-16+32=0$

$(g-4)^2=-16$

$(g-4)^2=16i^2$

$g-4=4i \;\vee\; g-4=-4i$

$g=4+4i \;\vee\; g=4-4i$.

$|4+4i|=4\sqrt{2}$ en $\arg(4-4i)=\frac{1}{4}\pi$.

Dus $u_n=\left(A\cos(\frac{1}{4}\pi n)+B\sin(\frac{1}{4}\pi n)\right)\cdot\left(4\sqrt{2}\right)^n$.

$u_0=5\Rightarrow(A\cdot 1+B\cdot 0)\cdot 1=A=5$ **(1)**

$u_1=28\Rightarrow\left(A\cdot\frac{1}{2}\sqrt{2}+B\cdot\frac{1}{2}\sqrt{2}\right)\cdot 4\sqrt{2}=28$ ofwel

$A\cdot\frac{1}{2}\sqrt{2}+4B=28\Rightarrow A+B=7$ **(2)**

(1) in **(2)** $\Rightarrow 5+B=7\Rightarrow B=2$.

Dus $u_n=\left(5\cos(\frac{1}{4}\pi n)+2\sin(\frac{1}{4}\pi n)\right)\cdot\left(4\sqrt{2}\right)^n$.

G39a In $u_n=-u_{n-1}-u_{n-2}+6$ **(1)** (geldt voor elke n) mag n vervangen worden door $n-1$. Dit geeft $u_{n-1}=-u_{n-2}-u_{n-3}+6$ **(2)**.

(2) in **(1)** $\Rightarrow u_n=-(-u_{n-2}-u_{n-3}+6)-u_{n-2}+6$

$u_n=u_{n-2}+u_{n-3}-6-u_{n-2}+6$

$u_n=u_{n-3}$. $(A=0,\ B=0$ en $C=1)$

G39b Substitueer $u_n=g^n$ in $u_n=u_{n-3}$.

$g^n=g^{n-3}$

$g^3=1$

$g^3-1=0$ (karakteristieke vergelijking)

$g^3=1=e^{k\cdot 2\pi i}$

$g=e^{k\cdot\frac{2}{3}\pi i}$

$g=1 \;\vee\; g=e^{\frac{2}{3}\pi i} \;\vee\; g=e^{\frac{4}{3}\pi i}$.

G39c $u_n = A\cdot 1^n + B\cdot \sin(\frac{2}{3}\pi n) + C\cdot\cos(\frac{2}{3}\pi n)$

$u_0 = 3 \Rightarrow A\cdot 1 + B\cdot 0 + C\cdot 1 = A + C = 3$ **(3)**

$u_1 = 1\frac{1}{2} \Rightarrow A\cdot 1 + B\cdot \frac{1}{2}\sqrt{3} + C\cdot -\frac{1}{2} = 1\frac{1}{2} \Rightarrow 2A + B\sqrt{3} - C = 3$ **(4)**

(1) $\Rightarrow u_2 = -1\frac{1}{2} - 3 + 6 = 1\frac{1}{2} \Rightarrow A\cdot 1 + B\cdot -\frac{1}{2}\sqrt{3} + C\cdot -\frac{1}{2} = 1\frac{1}{2} \Rightarrow 2A - B\sqrt{3} - C = 3$ **(5)**

$\begin{cases} 2A + B\sqrt{3} - C = 3 & \textbf{(4)} \\ 2A - B\sqrt{3} - C = 3 & \textbf{(5)}\ - \end{cases}$
$2B\sqrt{3} = 0$
$\underline{\underline{B = 0}}$

$\begin{cases} 2A + B\sqrt{3} - C = 3 & \textbf{(4)} \\ 2A - B\sqrt{3} - C = 3 & \textbf{(5)}\ + \end{cases}$
$4A - 2C = 6$
$2A - C = 3$ **(6)**

$\begin{cases} 2A - C = 3 & \textbf{(6)} \\ A + C = 3 & \textbf{(3)}\ + \end{cases}$
$3A = 6$
$\underline{\underline{A = 2}}$ in **(3)** $\Rightarrow 2 + C = 3 \Rightarrow \underline{\underline{C = 1}}.$

G40a $f(x) = \ln(1+x) \Rightarrow f'(x) = \frac{1}{1+x} \Rightarrow f''(x) = -\frac{1}{(1+x)^2} \Rightarrow f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f''''(x) = -\frac{6}{(1+x)^4} \Rightarrow f'''''(x) = \frac{24}{(1+x)^5}.$

G40b $f^{(1)}(x) = \frac{(-1)^{1-1}\cdot(1-1)!}{(1+x)^1} = \frac{(-1)^0\cdot 0!}{1+x} = \frac{1\cdot 1}{1+x} = \frac{1}{1+x};\quad f^{(2)}(x) = \frac{(-1)^{2-1}\cdot(2-1)!}{(1+x)^2} = \frac{(-1)^1\cdot 1!}{(1+x)^2} = \frac{-1\cdot 1}{(1+x)^2} = -\frac{1}{(1+x)^2};$

$f^{(3)}(x) = \frac{(-1)^{3-1}\cdot(3-1)!}{(1+x)^3} = \frac{(-1)^2\cdot 2!}{(1+x)^3} = \frac{1\cdot 2}{(1+x)^3} = \frac{2}{(1+x)^3};\quad f^{(4)}(x) = \frac{(-1)^{4-1}\cdot(4-1)!}{(1+x)^4} = \frac{(-1)^3\cdot 3!}{(1+x)^4} = \frac{-1\cdot 6}{(1+x)^4} = -\frac{6}{(1+x)^4};$

$f^{(5)}(x) = \frac{(-1)^{5-1}\cdot(5-1)!}{(1+x)^5} = \frac{(-1)^4\cdot 4!}{(1+x)^5} = \frac{1\cdot 24}{(1+x)^5} = \frac{24}{(1+x)^5};\quad \ldots\ldots\ldots\ldots\quad f^{(n)}(x) = \frac{(-1)^{n-1}\cdot(n-1)!}{(1+x)^n}.$

G40c $f(0) = \ln(1+0) = 0;\ f'(0) = \frac{1}{1+0} = 1;\ f''(0) = -\frac{1}{(1+0)^2} = -1;\ f'''(0) = \frac{2}{(1+0)^3} = 2;$

$f''''(0) = -\frac{6}{(1+0)^4} = -6;\ f'''''(x) = \frac{24}{(1+0)^5} = 24;\quad \ldots\ldots\ldots\ldots\quad f^{(n)}(0) = \frac{(-1)^{n-1}\cdot(n-1)!}{(1+0)^n} = (-1)^{n-1}\cdot(n-1)!.$

G40d $f(x) = f(0) + f'(0)\cdot x + f''(0)\cdot\frac{x^2}{2!} + f'''(0)\cdot\frac{x^3}{3!} + f''''(0)\cdot\frac{x^4}{4!} + f'''''(0)\cdot\frac{x^5}{5!} + \ldots$ (de formule van Maclaurin) geeft

$\ln(1+x) = 0 + 1\cdot x - 1\cdot\frac{x^2}{2!} + 2\cdot\frac{x^3}{3!} - 6\cdot\frac{x^4}{4!} + 24\cdot\frac{x^5}{5!} + \ldots$

$\ln(1+x) = 0 + x - \frac{x^2}{2} + 2\cdot\frac{x^3}{6} - 6\cdot\frac{x^4}{24} + 24\cdot\frac{x^5}{120} + \ldots$

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \ldots$

G40e $x = 1$ geeft $\ln(1+1) = 1 - \frac{1}{2}\cdot 1^2 + \frac{1}{3}\cdot 1^3 - \frac{1}{4}\cdot 1^4 + \frac{1}{5}\cdot 1^5 - \frac{1}{6}\cdot 1^6 + \ldots$

$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

Dus $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots = \ln(2).$

G40f $\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \ldots$

$z = -2$ geeft $\ln(1+-2) = \ln(-1) = -2 - \frac{1}{2}\cdot(-2)^2 + \frac{1}{3}\cdot(-2)^3 - \frac{1}{4}\cdot(-2)^4 + \frac{1}{5}\cdot(-2)^5 - \frac{1}{6}\cdot(-2)^6 + \ldots$

$= -2 - \frac{1}{2}\cdot 4 + \frac{1}{3}\cdot -8 - \frac{1}{4}\cdot 16 + \frac{1}{5}\cdot -32 - \frac{1}{6}\cdot 64 + \ldots$

$= -2 - 2 - \frac{8}{3} - 4 - \frac{32}{5} - \frac{32}{3}\cdot 64 - \ldots$

Er wordt een steeds groter getal afgetrokken, dus de reeks gaat naar min-oneindig.
Dus de reeks $\ln(1+z)$ is niet gedefinieerd voor $z = -2$.

G40g $z = i$ geeft $\ln(1+i) = i - \frac{1}{2}\cdot i^2 + \frac{1}{3}\cdot i^3 - \frac{1}{4}\cdot i^4 + \frac{1}{5}\cdot i^5 - \frac{1}{6}\cdot i^6 + \ldots$

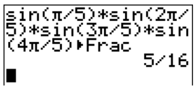
$|1+i| = \sqrt{2}$ en $\arg(1+i) = \frac{1}{4}\pi \Rightarrow (1+i) = \sqrt{2}\cdot e^{\frac{1}{4}\pi i}.$

Dit geeft $\ln(\sqrt{2}\cdot e^{\frac{1}{4}\pi i}) = i - \frac{1}{2}i^2 + \frac{1}{3}i^3 - \frac{1}{4}i^4 + \frac{1}{5}i^5 - \frac{1}{6}i^6 + \frac{1}{7}i^7 - \frac{1}{8}i^8 + \frac{1}{9}i^9 - \ldots$

$\ln(\sqrt{2}) + \ln(e^{\frac{1}{4}\pi i}) = i + \frac{1}{2} - \frac{1}{3}\cdot i - \frac{1}{4} + \frac{1}{5}\cdot i + \frac{1}{6} - \frac{1}{7}i - \frac{1}{8} + \frac{1}{9}i + \ldots$

$\frac{1}{2}\ln(2)\frac{1}{2} + \frac{1}{4}\pi i = (\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \ldots) + (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots)i$

Hieruit volgt $\frac{1}{4}\pi i = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$



G41a Zie het GR-scherm hiernaast.

G41b De oplossingen van de vergelijking $z^5 = 1$ zijn de nulpunten van de functie $f(z) = z^5 - 1$.

$z^5 = 1 = e^{k\cdot 2\pi i}$

$z = e^{k\cdot\frac{2}{5}\pi i}$

$k = 0, 1, 2, 3, 4$ geeft $z = 1 \vee z = e^{\frac{2}{5}\pi i} \vee z = e^{\frac{4}{5}\pi i} \vee z = e^{\frac{6}{5}\pi i} \vee z = e^{\frac{8}{5}\pi i}.$

Dus $z^5 - 1$ is als volgt te ontbinden: $z^5 - 1 = (z-1)(z - e^{\frac{2}{5}\pi i})(z - e^{\frac{4}{5}\pi i})(z - e^{\frac{6}{5}\pi i})(z - e^{\frac{8}{5}\pi i}).$

G41c Maak de staartdeling hiernaast of:

$(z-1)(z^4+z^3+z^2+z+1)=z^5+z^4+z^3+z^2+z-z^4-z^3-z^2-z-1=z^5-1.$

$z-1/z^5 \quad -1 \backslash z^4+\ldots$
$\underline{z^5-z^4} \; -$
$z^4 \quad -1$
$\ldots$

G41d Uit G41b en G41c volgt dat (voor elke z) geldt

$z^5-1=(z-1)(z-e^{\frac{2}{5}\pi i})(z-e^{\frac{4}{5}\pi i})(z-e^{\frac{6}{5}\pi i})(z-e^{\frac{8}{5}\pi i})=(z-1)(z^4+z^3+z^2+z+1).$

Dus $(z-e^{\frac{2}{5}\pi i})(z-e^{\frac{4}{5}\pi i})(z-e^{\frac{6}{5}\pi i})(z-e^{\frac{8}{5}\pi i})=z^4+z^3+z^2+z+1.$

$z=1$ geeft: $(1-e^{\frac{2}{5}\pi i})(1-e^{\frac{4}{5}\pi i})(1-e^{\frac{6}{5}\pi i})(1-e^{\frac{8}{5}\pi i})=1^4+1^3+1^2+1+1=5.$

Ook geldt $z^5-1=(z-1)(z-e^{-\frac{2}{5}\pi i})(z-e^{-\frac{4}{5}\pi i})(z-e^{-\frac{6}{5}\pi i})(z-e^{-\frac{8}{5}\pi i})$, want

$z^5=1=e^{k\cdot 2\pi i} \Rightarrow z=e^{k\cdot\frac{2}{5}\pi i}$ (met $k=0,-1,-2,-3,-4$) $\Rightarrow z=1 \vee z=e^{-\frac{2}{5}\pi i} \vee z=e^{-\frac{4}{5}\pi i} \vee z=e^{-\frac{6}{5}\pi i} \vee z=e^{-\frac{8}{5}\pi i}.$

Dus geldt ook: $(1-e^{-\frac{2}{5}\pi i})(1-e^{-\frac{4}{5}\pi i})(1-e^{-\frac{6}{5}\pi i})(1-e^{-\frac{8}{5}\pi i})=5.$

G41e $(1-e^{\frac{2}{5}\pi i})\cdot(1-e^{-\frac{2}{5}\pi i})=1-e^{-\frac{2}{5}\pi i}-e^{\frac{2}{5}\pi i}+e^0=1-(e^{-\frac{2}{5}\pi i}+e^{\frac{2}{5}\pi i})+1=2-2\cdot\frac{e^{\frac{2}{5}\pi i}+e^{-\frac{2}{5}\pi i}}{2}=2-2\cos(\frac{2}{5}\pi).$

$2-2\cos(\frac{2}{5}\pi)=2-2\cos(2\cdot\frac{1}{5}\pi)=2-2\left(1-2\sin^2(\frac{1}{5}\pi)\right)=2-2+4\sin^2(\frac{1}{5}\pi)=4\sin^2(\frac{1}{5}\pi)$ (gebruik $\cos(2A)=1-2\sin^2(A)$)

G41f $\underbrace{(1-e^{\frac{2}{5}\pi i})(1-e^{\frac{4}{5}\pi i})(1-e^{\frac{6}{5}\pi i})(1-e^{\frac{8}{5}\pi i})}_{=5 \text{ (zie G41d)}}\underbrace{(1-e^{-\frac{2}{5}\pi i})(1-e^{-\frac{4}{5}\pi i})(1-e^{-\frac{6}{5}\pi i})(1-e^{-\frac{8}{5}\pi i})}_{=5 \text{ (zie G41d)}}=5\cdot 5=25$

$\underbrace{(1-e^{\frac{2}{5}\pi i})(1-e^{-\frac{2}{5}\pi i})}_{=4\sin^2(\frac{1}{5}\pi) \text{ (zie G41e)}}\underbrace{(1-e^{\frac{4}{5}\pi i})(1-e^{-\frac{4}{5}\pi i})}_{=4\sin^2(\frac{2}{5}\pi)}\underbrace{(1-e^{\frac{6}{5}\pi i})(1-e^{-\frac{4}{5}\pi i})}_{=4\sin^2(\frac{3}{5}\pi)}\underbrace{(1-e^{\frac{8}{5}\pi i})(1-e^{-\frac{8}{5}\pi i})}_{=4\sin^2(\frac{4}{5}\pi)}=25$

$4^4\cdot\left(\sin(\frac{\pi}{5})\cdot\sin(\frac{2\pi}{5})\cdot\sin(\frac{3\pi}{5})\cdot\sin(\frac{4\pi}{5})\right)^2=25$

$\left(\sin(\frac{\pi}{5})\cdot\sin(\frac{2\pi}{5})\cdot\sin(\frac{3\pi}{5})\cdot\sin(\frac{4\pi}{5})\right)^2=\frac{5^2}{4^4}$

$\sin(\frac{\pi}{5})\cdot\sin(\frac{2\pi}{5})\cdot\sin(\frac{3\pi}{5})\cdot\sin(\frac{4\pi}{5})=\frac{5}{4^2}=\frac{5}{16}.$

G41g $z^m=1$ (m geheel en positief) $\Rightarrow z=e^{k\cdot 2\pi i} \Rightarrow z=e^{k\cdot\frac{2\pi}{m}i}$

Dus $z=1 \vee z=e^{\frac{2\pi}{m}i} \vee z=e^{\frac{4\pi}{m}i} \vee z=e^{\frac{6\pi}{m}i} \vee \ldots \vee z=e^{\frac{(m-1)2\pi}{m}i}.$

Maar ook $z=1 \vee z=e^{-\frac{2\pi}{m}i} \vee z=e^{-\frac{4\pi}{m}i} \vee z=e^{-\frac{6\pi}{m}i} \vee \ldots \vee z=e^{-\frac{(m-1)2\pi}{m}i}.$

Hieruit volgt: $z^m-1=(z-1)(z-e^{\frac{2\pi}{m}i})(z-e^{\frac{4\pi}{m}i})(z-e^{\frac{6\pi}{m}i})\cdot\ldots\cdot(z-e^{\frac{(m-1)2\pi}{m}i})$

en $z^m-1=(z-1)(z-e^{-\frac{2\pi}{m}i})(z-e^{-\frac{4\pi}{m}i})(z-e^{-\frac{6\pi}{m}i})\cdot\ldots\cdot(z-e^{-\frac{(m-1)2\pi}{m}i}).$

Verder geldt: $z^m-1=(z-1)(\underbrace{z^{m-1}+z^{m-2}+z^{m-3}+\ldots+1}_{m \text{ termen}}).$

Uit bovenstaande volgt:

$\underbrace{(1-e^{\frac{2\pi}{m}i})(1-e^{\frac{4\pi}{m}i})(1-e^{\frac{6\pi}{m}i})\cdot\ldots\cdot(1-e^{\frac{(m-1)2\pi}{m}i})}_{=m}\cdot\underbrace{(1-e^{-\frac{2\pi}{m}i})(1-e^{-\frac{4\pi}{m}i})(1-e^{-\frac{6\pi}{m}i})\cdot\ldots\cdot(1-e^{-\frac{(m-1)2\pi}{m}i})}_{=m}=m^2$

$\underbrace{(1-e^{\frac{2\pi}{m}i})(1-e^{-\frac{2\pi}{m}i})}_{=4\sin^2(\frac{\pi}{m})}\underbrace{(1-e^{\frac{4\pi}{m}i})(1-e^{-\frac{4\pi}{m}i})}_{=4\sin^2(\frac{2\pi}{m})}\underbrace{(1-e^{\frac{6\pi}{m}i})(1-e^{-\frac{6\pi}{m}i})}_{=4\sin^2(\frac{3\pi}{m})}\cdot\ldots\cdot\underbrace{(1-e^{\frac{(m-1)2\pi}{m}i})(1-e^{-\frac{(m-1)2\pi}{m}i})}_{=4\sin^2(\frac{(m-1)\pi}{m})}=m^2$

$4^{m-1}\cdot\left(\sin(\frac{\pi}{m})\cdot\sin(\frac{2\pi}{m})\cdot\sin(\frac{3\pi}{m})\cdot\ldots\cdot\sin(\frac{(m-1)\pi}{m})\right)^2=m^2$

$\left(2^2\right)^{m-1}\cdot\left(\sin(\frac{\pi}{m})\cdot\sin(\frac{2\pi}{m})\cdot\sin(\frac{3\pi}{m})\cdot\ldots\cdot\sin(\frac{(m-1)\pi}{m})\right)^2=m^2$

$\left(2^{m-1}\right)^2\cdot\left(\sin(\frac{\pi}{m})\cdot\sin(\frac{2\pi}{m})\cdot\sin(\frac{3\pi}{m})\cdot\ldots\cdot\sin(\frac{(m-1)\pi}{m})\right)^2=m^2$

$\left(\sin(\frac{\pi}{m})\cdot\sin(\frac{2\pi}{m})\cdot\sin(\frac{3\pi}{m})\cdot\ldots\cdot\sin(\frac{(m-1)\pi}{m})\right)^2=\frac{m^2}{\left(2^{m-1}\right)^2}$

$\sin(\frac{\pi}{m})\cdot\sin(\frac{2\pi}{m})\cdot\sin(\frac{3\pi}{m})\cdot\ldots\cdot\sin(\frac{(m-1)\pi}{m})=\frac{m}{2^{m-1}}.$